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THE

MESSENGER OF MATHEMATICS.

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CONTENTS.

ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.

PAGE

On the definition of an invariant. By *E. B. Elliot* - - - - 5

Note on a class of algebraical identities. By *R. Lachlan* - - - 21

Note on the sine-equation in spherical trigonometry. By *M. Jenkins* - 37

On double algebra. By *A. Buchheim* - - - - - 62

Note on triple algebra. By *A. Buchheim* - - - - - 111

The 'sine-triple-angle' circle. By *R. Tucker* - - - - 125

Définition d'un invariant. By Prof. *Paul Mansion* - - - - 127

Note on the binomial-theorem coefficients. By *A. Gutzmer* - - 190

GEOMETRY OF TWO AND THREE DIMENSIONS.

On the order of orthoptic loci. By *C. Taylor* - - - - 1

On the application of the theory of complex quantities to plane geometry. By *J. Brill* - - - - - - - - 8

Note on the cyclide. By *A. R. Johnson* - - - - - 33

On a geometrical interpretation of the algebraical expression which, equated to zero, represents a curve or a surface. By *H. M. Taylor* - - 39

Note on the quadric and the cubic. By *A. R. Johnson* - - - 63

Orthogonal systems of circles. By *R. Lachlan* - - - - 98

Proof of the formula for the torsion of a geodesic. By *G. S. Carr* - - 110

A proof of Holditch's theorem. By *M. Jenkins* - - - - 110

On conics satisfying given conditions and touching a given conic. By *R. Lachlan* - - - - - - - - 140

Extension of an inversion property. By *H. M. Taylor* - - - 143

On poristic systems of circles. By *R. Lachlan* - - - - 152

Some properties of confocal conics and a derived cubic. By *F. Morley* - 181

DIFFERENTIAL AND INTEGRAL CALCULUS AND DIFFERENTAL EQUATIONS.

PAGE

A proof of Fourier's series for periodic functions. By *P. Alexander* - 23

On Cayley's differential equation for orthogonal surfaces. By *A. R. Johnson* - - - - - - - - 27, 111

Extensions of Fourier's trigonometric series theorems. By *P. Alexander* - 42

On a point connected with symbolic methods of integration. By Prof. *W. Woolsey Johnson* - - - - - - - 86

A proof of Fourier's series theorem. By *A. R. Johnson* - - - 90

On the solution of Legendre's equation in a particular case. By *A. R. Forsyth* - - - - - - - - 163

Note on the singular solutions, &c., of homogeneous differential equations. By Prof. *W. Woolsey Johnson* - - - - - - 186

THEORY OF ELLIPTIC FUNCTIONS.

Note on theorems in Weierstrass's theory of elliptic functions. By *A. Buchheim* - - - - - - - - 35

Formulæ in elliptic functions. By *J. W. L. Glaisher* - - - 67

Comparison of the Weierstrassian and Jacobian elliptic functions. By Prof. *Cayley* - - - - - - - 129

On the deduction of the q-series for the elliptic functions from the q-products. By *J. W. L. Glaisher* - - - - - 133

On the process of squaring the q-series for $k\rho$ sn u, $k\rho$ cn u, ρ dn u. By *J. W. L. Glaisher* - - - - - - - 145

On the process of squaring the Zeta functions. By *J. W. L. Glaisher* - 150

On the trisection of the periods for Weierstrass's elliptic functions. By *W. Burnside* - - - - - - - - 177

Squaring ρ dn u. By *N. M. Ferrers* - - - - - - 189

APPLIED MATHEMATICS.

The period equation for lateral vibrations. By *A. G. Greenhill* - - 115

Summer lines on Mercator's chart. By *A. G. Greenhill* - - - 162

MESSENGER OF MATHEMATICS.

ON THE ORDER OF ORTHOPTIC LOCI.

By *C. Taylor, D.D.*

If to any curve there be drawn a pair of tangents at right angles to one another, the locus of their point of concourse may be called its *Orthoptic Locus*, since at every point thereof the curve subtends or is seen under a right angle.

The first step towards a theory of such loci was the determination of the orthoptic locus of a conic. This was shewn by De la Hire to be a circle, which in the case of the parabola degenerates into a straight line—or rather into a line-circle, consisting of the directrix and the line at infinity. One of the ways in which this result may be arrived at is by finding in what points the locus meets the line at infinity; and the same method will be seen to be applicable to curves of all classes. The proof depends upon a peculiarity of the two imaginary points at infinity through which all circles in a plane pass.

1. *The circular points at infinity.*

Straight lines which meet at infinity are in all cases parallel, and in general cannot be regarded as inclined at any finite angle. But an exception must be made in the case of the two imaginary points I and J at infinity through which all circles pass.

(*a*) For through one of them I draw any two lines IA and IB. Then the points A and B subtend any assumed angle α at all points on the circumference of a certain circle, and therefore at I. Therefore IA and IB, although they meet at infinity, may nevertheless be regarded as including the angle α, which may be of any magnitude whatsoever. That is to say, the angle between any two lines drawn through one of the circular points is *indeterminate*, and may be said to be of any assumed magnitude.

Hence, by supposing the points A and B to coincide, we see that any straight line IA drawn through a circular point makes *an indeterminate angle with itself.*

It is commonly said that such a line is at right angles to itself. But the statement should not be so limited. For the tangent of the angle between the lines $y-mx=0$, and $y-m'x=0$, is $\frac{m-m'}{1+mm'}$, so that when $m=m'=\sqrt{-1}$, not only does the denominator vanish, but the numerator also. The angle which any line $y=\sqrt{-1}\,.\,x+c$ makes with itself is therefore indeterminate.

(*b*) The existence of the circular points may be proved as follows.

On a given circle take points A and B. These subtend an angle α at the circumference, and therefore at either of the points in which the circle meets the line at infinity. Call these I and J, and draw IA' and IB' to any points A' and B' in the plane.

Then since I is at infinity, the angle AIA' is equal to zero,* and likewise the angle BIB'. Therefore the angle $A'IB'$ is equal to AIB, or α, which may be of any magnitude, according to the assumed positions of the points A and B on the circle.

That is to say, any two points A' and B' in the plane subtend an indeterminate angle at I; and therefore all circles in the plane pass through I, and in like manner through J; and no finite circle can pass through more than these two points on the line at infinity.

1. *The ellipse.*

If two tangents be drawn from I or J to any curve, they may be said to be at right angles. These two points therefore belong to the orthoptic locus of the curve; and in general no other points at infinity belong to it.

(*a*) The ellipse being a curve of the second class, two tangents only can be drawn to it from either of the circular points. These are therefore *single* points on the orthoptic locus, and the locus is accordingly a circle.

At the points in which this locus cuts the ellipse the tangent to the ellipse must be at right angles to itself, and and must therefore pass through I or J. That is to say, it must be a tangent from one of the foci, and must therefore have its point of contact on the corresponding directrix, the polar of the focus. The orthoptic locus is therefore *the circle through the points in which the ellipse cuts its directrices.*

---

* An angle subtended at a point on the line at infinity may always be said to be equal to zero, or to have zero for one of its values.

(*b*) Since SI and SJ are tangents, each at right angles to itself, these two lines, which make up the point-circle at the focus S, might be said to belong, in a special sense, to the orthoptic locus. But in this theory the degree of the locus is estimated independently of such factors.

If two tangents at right angles have for their equations,

$$y - mx = \sqrt{(b^2 + m^2a^2)},$$

and

$$my + x = \sqrt{(m^2b^2 + a^2)},$$

the equation to their locus of intersection will be,

$$(1 + m^2)(x^2 + y^2 - a^2 - b^2) = 0,$$

where the $(1 + m^2)$ points to the tangents from the foci.

2. *The cardioid.*

It is known that in the cardioid, according as the chord of contact of the orthogonal tangents does or does not pass through the cusp, their point of concourse traces a *circle* or a *bicircular quartic.*

These together make up a tricircular sextic; and such should be the orthoptic locus of any curve of the third class. For from I or J three tangents can be drawn to it; and these combine to make three quasi-orthogonal pairs of tangents. Each of the circular points is therefore a threefold point on the orthoptic locus, and these are in general its only points at infinity.

3. *Of orthoptic loci in general.*

To a curve of the n^{th} class n tangents can be drawn from I or J; and these may be taken in pairs in $n\,\dfrac{n-1}{2}$ ways; and every pair may be regarded as at right angles to one another.

Each of the circular points is therefore a point of the order $n\,\dfrac{n-1}{2}$ on the orthoptic locus; and they are in general its only points at infinity. Consequently the order or degree of the locus is $n(n-1)$. That is to say, *the orthoptic locus of any curve of class n is of the order* $n(n-1)$, *and it passes* $n\,\dfrac{n-1}{2}$ *times through the circular points.*

In verification of this result, notice that when the curve of class n degenerates into n points, its orthoptic locus evidently consists of the $n\,\dfrac{n-1}{2}$ circles which have the lines joining the points two and two for diameters. In this we omit the point-circles at the n points, which might have been regarded as factors of the locus.

4. *The parabola.*

Of the two tangents that can be drawn to a parabola from any point on the line at infinity one is that line itself, which may be regarded as orthogonal to every other tangent.

Every point on the line at infinity is therefore a single point on the orthoptic locus of the parabola; and the remaining factor of the locus must be a straight line not at infinity.

This straight line is the directrix, the polar of the focus; since each of the tangents to the parabola from the focus is at right angles to itself, and therefore has its point of contact on the orthoptic locus.

Conversely, given that the directrix is an orthoptic locus, the parabola must touch the line at infinity; for the tangents to it from the point at infinity on the directrix could only be regarded as parallel unless one of them were the line at infinity.

There is a corresponding reduction in the order of the orthoptic locus of a curve which has contact at more points than one with the line at infinity.

5. *The order of pedals.*

Given a curve of the n^{th} class and a point O, the locus of the foot of the perpendicular from O to a tangent to the curve will be an n-circular curve of the order $2n$.

For the line OI or OJ may be regarded as perpendicular to every one of the n tangents from I or J respectively. Each of the circular points is therefore an n-fold point on the pedal, and this (having no other points at infinity) is of the order $2n$.

When O coincides with a focus, each of the lines OI, OJ is a tangent, and is also perpendicular to itself. Consequently these lines, which make up the point-circle at O, belong to the locus; and its remaining factor is of the order $2(n-1)$.

For example, in the ellipse, in which any tangent and the perpendiculars upon it from a focus may be represented by the equations,

$$y - mx = \sqrt{(b^2 + m^2a^2)},$$

and

$$my + x = \sqrt{(a^2 - b^2)},$$

the equation to the pedal with respect to a focus is,

$$(1 + m^2)(x^2 + y^2 - a^2) = 0.$$

The pedal therefore (which is in general a bicircular quartic) breaks up in this case into the point-circle at the focus and the circle,

$$x^2+y^2=a^2.$$

If by an extension of the term pedal the locus of intersection of pairs of orthogonal tangents, drawn one to each of two curves, be called the *pedal* of the two curves, or of the one with respect to the other; then since from I or J there can be drawn m tangents to a curve of the m^{th} class, and n to a curve of the n^{th} class, the pedal of two such curves passes mn times through the circular points, and is of the order $2mn$.

This result may also be deduced from the formula for the orthoptic locus, as follows. The two curves of class m and class n respectively make up a curve of class $(m+n)$, whose orthoptic locus consists of the orthoptic loci and the pedal of the two curves. Their pedal is therefore of the order,

$$(m+n)(m+n-1)-m(m-1)-n(n-1),$$

that is to say, of the order $2mn$.

ON THE DEFINITION OF AN INVARIANT.

By *E. B. Elliott, M.A.*

1. THE usual definition as given in Salmon's and other treatises runs as follows:—"Any function of the coefficients of a quartic is called an invariant, if, when the quadric is linearly transformed, the same function of the new coefficients is equal to the old function *multiplied by some power of the modulus of transformation*; that is to say, when we have

$$\phi(A, B, C, \ldots)=\Delta^w\phi(a, b, c, \ldots)."$$

It is, I think, better to replace the words in Italics by others less exacting, viz.—"multiplied by some factor depending only on the coefficients in the scheme of transformation"—and to prove at once the important fact that this factor *of necessity must* have the special form Δ^w. This theorem it is in fact customary tacitly to assume at the outset: and it seems strange that a statement and proof of it should not be a usual introduction to the subject so that full rigour might be at once given to certain indirect processes of investigation. It is merely from desire to suggest what seems to me an improvement in form, and not from any notion that the great researchers on the subject have failed to notice so elementary a matter, that I venture to mention the proposition and append the following proof.

2. Let us denote by $l, m, \dots l', m', \dots$ the assemblage of coefficients in our scheme of linear transformation: let $a, b, c, \dots$ be the coefficients in the old form of a q-ary p-ic u, and $A, B, C, \dots$ the corresponding coefficients in the form given to it by the transformation. The expression of invariant of a function $\phi(a, b, c, \dots)$ is taken to be

$$\phi(A, B, C, \dots) = \psi(l, m, \dots l', m', \dots)\, \phi(a, b, c, \dots) \dots (1).$$

In this we must prove the homogeneity of ϕ and ψ. (The usual proof that the former is homogeneous is not admissible; for it assumes that the latter is a power of Δ, which it is our object to establish.)

Now partial differentiation in turn with regard to the q^2 quantities $l, m, \dots l', m', \dots$ gives us q^2 equations of the type

$$\frac{d\phi}{dA}\,\frac{dA}{dl} + \frac{d\phi}{dB}\,\frac{dB}{dl} + \dots = \frac{d\psi}{dl}\,\phi(a, b, c, \dots):$$

multiplying which by $l, m, \dots l', m', \dots$ respectively, adding, and remembering that $A, B, C \dots$ are homogeneous and of the p^{th} degree in $l, m, \dots l', m', \dots$, we obtain, by use of Euler's Theorem,

$$p\left\{A\frac{d\phi}{dA} + B\frac{d\phi}{dB} + \dots\right\} = \Sigma\left(l\frac{d\psi}{dl}\right)\phi(a, b, c, \dots),$$

$$= \Sigma\left(l\frac{d\psi}{dl}\right)\frac{\phi(A, B, C, \dots)}{\psi(l, m, \dots l', m', \dots)},$$

i.e.

$$\frac{\Sigma\left(A\frac{d\phi}{dA}\right)}{\phi(A, B, C, \dots)} = \frac{\Sigma\left(l\frac{d\psi}{dl}\right)}{p\psi(l, m, \dots l', m', \dots)},$$

in which equation the left-hand member contains only the first, and the right hand member only the second, of the two sets of quantities $A, B, C, \dots$ and $l, m, \dots l', m', \dots$. But these two sets of quantities can be chosen quite independently of one another. Each side of the equation is then a mere constant not depending on either set. Consequently, by the converse of Euler, both $\phi(A, B, C, \dots)$ and $\psi(l, m, \dots l', m', \dots)$ are homogeneous.

3. Now Δ being the modulus

$$\begin{vmatrix} l, & m, & \ldots \\ l', & m', & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

of the transformation,

$$x = l\,X + m\,Y + \ldots,$$
$$y = l'X + m'Y + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

which produced the transformed form of u, the reverse substitution

$$X = \frac{1}{\Delta}\left\{\frac{d\Delta}{dl}x + \frac{d\Delta}{dl'}y + \ldots\right\},$$

$$Y = \frac{1}{\Delta}\left\{\frac{d\Delta}{dm}x + \frac{d\Delta}{dm'}y + \ldots\right\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

must restore u to its original form. By application of (1) to this second transformation we must have, therefore,

$$\phi\,(a, b, c, \ldots)$$
$$= \psi\left(\frac{1}{\Delta}\frac{d\Delta}{dl}, \frac{1}{\Delta}\frac{d\Delta}{dl'}, \ldots \frac{1}{\Delta}\frac{d\Delta}{dm}, \frac{1}{\Delta}\frac{d\Delta}{dm'}, \ldots\right)\phi\,(A, B, C, \ldots)$$
$$= \frac{1}{\Delta^{kp}}\psi\left(\frac{d\Delta}{dl}, \frac{d\Delta}{dl'}, \ldots \frac{d\Delta}{dm}, \frac{d\Delta}{dm'}, \ldots\right)\phi\,(A, B, C, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

in virtue of the homogeneity of ψ, if k be the degree of ϕ and consequently kp that of ψ. We have then, in (1) and (2), two expressions for the ratio of $\phi\,(a, b, c, \ldots)$ to $\phi\,(A, B, C, \ldots)$ by equation of which we get

$$\psi\,(l, m, \ldots l', m', \ldots)\,\psi\left(\frac{d\Delta}{dl}, \frac{d\Delta}{dl'}, \ldots \frac{d\Delta}{dm}, \frac{d\Delta}{dm'}, \ldots\right) = \Delta^{kp}.$$

In this identity the right-hand member has no factor which is not a power of Δ, for Δ is irresoluble. Moreover the function ψ is rational and integral if the function ϕ be so—a limitation to the form of this latter which should be imposed in the definition. The two factors on the left must therefore be powers of Δ, or at worst constant multiples μ and μ^{-1} of such powers. Hence, remembering that Δ is of degree q in l, &c., and in consequence that $\frac{d\Delta}{dl}$ &c., are of degree $q-1$,

$$\psi(l, m, \dots l', m', \dots) = \Delta^{\frac{kp}{q}},$$

and
$$\psi\left(\frac{d\Delta}{dl}, \frac{d\Delta}{dl'}, \dots \frac{d\Delta}{dm}, \frac{d\Delta}{dm'}, \dots\right) = \Delta^{\frac{kq(q-1)}{q}},$$

no multiple μ being required, since the right-hand members are the $\frac{kp}{q}$th powers of Δ and its reciprocal determinant, which we know to be the same functions respectively of the two sets of arguments in the left hand members.

4. Like statements and analogous reasoning are applicable to covariants.

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ON THE APPLICATION OF THE THEORY OF COMPLEX QUANTITIES TO PLANE GEOMETRY.

By *J. Brill.*

1. Although the method of representing a complex quantity by means of a vector has been known for a considerable period of time, and has done substantial service to certain branches of mathematics, yet the application of the method to the investigation of the properties of plane figures seems to have attracted comparatively little attention. Now, as the multiplication of complex quantities obeys the commutative law, it is evident that all the ordinary algebraical identities are true when the quantities involved in them are complex. This being the case, if we take any algebraical identity and seek to obtain an interpretation of it by the method in question, we shall obtain a new geometrical theorem. A few specimens of this method of proceeding are given in the following paper. It will be seen that some of the results are easily obtainable by the older methods, while in some cases, notably the results in Arts. 8, 15 and 16, the application of the older methods would present considerable difficulty; and, in the great

majority of cases, it is evident that this method possesses the great merit of superior conciseness. Were we to seek to obtain a series of new methods of interpretation we should be rewarded with a multitude of new geometrical theorems.

2. As a first example we shall obtain an interpretation of the identity

$$x(y-z)+y(z-x)+z(x-y)=0.$$

Let ABC be a triangle (fig. 1), and O a point within it. Join OA, OB, OC, and let

$$(OA)=a(\cos\alpha+i\sin\alpha),\quad (OB)=b(\cos\beta+i\sin\beta),$$
$$(OC)=c(\cos\gamma+i\sin\gamma),$$
$$(BC)=d(\cos\delta+i\sin\delta),\quad (CA=e(\cos\varepsilon+i\sin\varepsilon),$$
$$(AB)=f(\cos\zeta+i\sin\zeta).$$

Now, we have

$$(BC)=(OC)-(OB),\ (CA)=(OA)-(OC),\ (AB)=(OB)-(OA);$$

and, consequently, by applying the above identity we obtain

$$(OA).(BC)+(OB).(CA)+(OC).(AB)=0.$$

From this it follows that

$$ad\cos(\alpha+\delta)+be\cos(\beta+\varepsilon)+cf\cos(\gamma+\zeta)=0,$$

and $$ad\sin(\alpha+\delta)+be\sin(\beta+\varepsilon)+cf\sin(\gamma+\zeta)=0.$$

Consequently

$$\frac{ad}{\sin(\beta+\varepsilon-\gamma-\zeta)}=\frac{be}{\sin(\gamma+\zeta-\alpha-\delta)}=\frac{cf}{\sin(\alpha+\delta-\beta-\varepsilon)}.$$

Now, referring to the figure, we see that

$$\beta-\gamma=2\pi-BOC,\ \text{and}\ \varepsilon-\zeta=BAC-\pi;$$

therefore $$\beta+\varepsilon-\gamma-\zeta=\pi-(BOC-BAC),$$

and $$\sin(\beta+\varepsilon-\gamma-\zeta)=\sin(BOC-BAC).$$

Similarly we can prove that

$$\sin(\gamma+\zeta-\alpha-\delta)=\sin(COA-CBA),$$

and $$\sin(\alpha+\delta-\beta-\varepsilon)=\sin(AOB-ACB).$$

Thus we obtain

$$\frac{OA\,.\,BC}{\sin(BOC-BAC)}=\frac{OB\,.\,CA}{\sin(COA-CBA)}=\frac{OC\,.\,AB}{\sin(AOB-ACB)}\,;$$

3. The same figure will also furnish us with interpretations of the two well known identities

$$yz\,(y-z)+zx\,(z-x)+xy\,(x-y)+(y-z)\,(z-x)\,(x-y)=0,$$

and

$$x^2\,(y-z)+y^2\,(z-x)+z^2\,(x-y)+(y-z)\,(z-x)\,(x-y)=0.$$

Thus we have

$$(OB).(OC).(BC)+(OC).(OA).(CA)$$
$$+(OA).(OB).(AB)+(BC).(CA).(AB)=0,$$

and

$$(OA)^2.(BC)+(OB)^2.(CA)+(OC)^2.(AB)$$
$$+(BC).(CA).(AB)=0.$$

It follows from the first of these results that

$$bcd\cos(\beta+\gamma+\delta)+cae\cos(\gamma+\alpha+\varepsilon)$$
$$+abf\cos(\alpha+\beta+\zeta)+def\cos(\delta+\varepsilon+\zeta)=0,$$

and

$$bcd\sin(\beta+\gamma+\delta)+cae\sin(\gamma+\alpha+\varepsilon)$$
$$+abf\sin(\alpha+\beta+\zeta)+def\sin(\delta+\varepsilon+\zeta)=0,$$

whence we obtain

$$bcd\cos(\beta+\gamma-\varepsilon-\zeta)+cae\cos(\gamma+\alpha-\zeta-\delta)$$
$$+abf\cos(a+\beta-\delta-\varepsilon)+def=0.$$

Now $\varepsilon-\gamma=\pi-OCA$ and $\zeta-\beta=OBA$,

therefore $\beta+\gamma-\varepsilon-\zeta=-(\pi+OBA-OCA)$

and

$$\cos(\beta+\gamma-\varepsilon-\zeta)=\cos(\pi+OBA-OCA)=-\cos(OBA-OCA).$$

Similarly we can prove that

$$\cos(\gamma+\alpha-\zeta-\delta)=-\cos(OCB-OAB)$$

and $\cos(\alpha+\beta-\delta-\varepsilon)=-\cos(OAC-OBC).$

Hence we obtain

$$OB.OC.BC\cos(OBA-OCA)+OC.OA.CA\cos(OCB-OAB)$$
$$+OA.OB.AB\cos(OAC-OBC)=BC.CA.AB.$$

Interpreting the second result in a similar manner we obtain

$$OA^2.BC\cos(OAB-OAC)+OB^2.CA\cos(OBC-OBA)$$
$$+OC^2.AB\cos(OCA-OCB)=BC.CA.AB.$$

4. Utilizing the same figure to obtain an interpretation of the identity

$$x^3(y-z)+y^3(z-x)+z^3(x-y)+(x+y+z)(y-z)z-x)(x-y)=0,$$

we arrive at the following:—

Let ABC be a triangle, P its centroid, and O any other point within it, then

$$OA^3.BC\cos(OAB-OAC+POA)$$
$$+OB^3.CA\cos(OBC-OBA+POB)$$
$$+OC^3.AB\cos(OCA-OCB+POC)=3.BC.CA.AB.OP.$$

In this result the angles POA, POB, POC are all supposed to be measured in the same sense, the same as that in which BOC, COA, AOB are measured.

Similarly as an interpretation of the identity

$$x(y-z)^3+y(z-x)^3+z(x-y)^3=(y-z)(z-x)(x-y)(x+y+z)$$

we obtain the result

$$OA.BC^3\cos(C-B+POA)+OB.CA^3\cos(A-C+POB)$$
$$+OC.AB^3\cos(B-A+POC)=3.BC.CA.AB.OP,$$

the same stipulation being introduced with respect to the direction of measurement of the angles POA, POB, POC.

5. We will give one more interpretation by means of this figure, namely that of the identity

$$x^2(y+z)+y^2(z+x)+z^2(x+y)$$
$$=yz(y+z)+zx(z+x)+xy(x+y).$$

Let ABC be a triangle, D, E, F the middle points of its sides, and O any point within it. Then we have

$$\begin{aligned} &OA^2.\,OD\cos(AOE-AOF)+OB^2.\,OE\cos(BOF-BOD)\\ &+OC^2.\,OF\cos(COD-COE)=OB.OC.OD\cos(BOF-COE)\\ &+OC.OA.OE\cos(COD-AOF)\\ &\qquad\qquad\qquad+AO.OB.OF\cos(AOE-BOD). \end{aligned}$$

6. Let ABC be a triangle (fig. 2), and let

$$(BC)=a(\cos\alpha+i\sin\alpha),\quad (CA)=b(\cos\beta+i\sin\beta),$$
$$(AB)=c(\cos\gamma+i\sin\gamma).$$

We have $(BC)+(CA)+(AB)=0$, and consequently

$$(BC)^3+(CA)^3+(AB)^3=3.(BC).(CA).(AB).$$

Hence

$$a^3\cos 3\alpha+b^3\cos 3\beta+c^3\cos 3\gamma=3abc\cos(\alpha+\beta+\gamma)$$

and $\quad a^3\sin 3\alpha+b^3\sin 3\beta+c^3\sin 3\gamma=3abc\sin(\alpha+\beta+\gamma),$

consequently

$$a^3\cos(\beta+\gamma-2\alpha)+b^3\cos(\gamma+\alpha-2\beta)+c^3\cos(\alpha+\beta-2\gamma)=3abc.$$

Now $\quad \beta-\alpha=\pi-C$ and $\gamma-\alpha=\pi+B,$

consequently $\quad \beta+\gamma-2\alpha=2\pi+B-C$

and $\quad \cos(\beta+\gamma-2\alpha)=\cos(B-C).$

Similarly $\quad \cos(\gamma+\alpha-2\beta)=\cos(C-A)$

and $\quad \cos(\alpha+\beta-2\gamma)=\cos(A-B);$

thus we obtain the result

$$a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc.$$

7. If $\quad x+y+z=0$ and $\xi+\eta+\zeta=0,$

we have $\quad \eta z-y\zeta=\zeta x-z\xi=\xi y-x\eta,$

or $\quad \dfrac{\eta}{y}-\dfrac{\zeta}{z}:\dfrac{\zeta}{z}-\dfrac{\xi}{x}:\dfrac{\xi}{x}-\dfrac{\eta}{y}::x:y:z.$

Now let ABC and $A'B'C'$ be two triangles, (fig. 3), and let

$$(BC)=a(\cos\alpha+i\sin\alpha),\quad (CA)=b(\cos\beta+i\sin\beta),$$
$$(AB)=c(\cos\gamma+i\sin\gamma),$$
$$(B'C')=a'(\cos\alpha'+i\sin\alpha'),\quad (C'A')=b'(\cos\beta'+i\sin\beta'),$$
$$(A'B')=c'(\cos\gamma'+i\sin\gamma').$$

We have $$(BC)+(CA)+(AB)=0$$
and $$(B'C')+(C'A')+(A'B')=0,$$
and consequently
$$\frac{(C'A')}{(CA)}-\frac{(A'B')}{(AB)} : \frac{(A'B')}{(AB)}-\frac{(B'C')}{(BC)} : \frac{(B'C')}{(BC)}-\frac{(C'A')}{(CA)}$$
$$:: (BC):(CA):(AB).$$

These six complexes being proportional, it follows that their moduli must be proportional also.

Now
$$\frac{(C'A')}{(CA)}-\frac{(A'B')}{(AB)}=\frac{b'}{b}\left\{\cos(\beta'-\beta)+i\sin(\beta'-\beta)\right\}$$
$$-\frac{c'}{c}\left\{\cos(\gamma'-\gamma)+i\sin(\gamma'-\gamma)\right\}$$
$$=\frac{b'}{b}\cos(\beta'-\beta)-\frac{c'}{c}\cos(\gamma'-\gamma)$$
$$+i\left\{\frac{b'}{b}\sin(\beta'-\beta)-\frac{c'}{c}\sin(\gamma'-\gamma)\right\};$$

hence the square of its modulus is equal to
$$\left(\frac{b'}{b}\right)^2+\left(\frac{c'}{c}\right)^2-2\frac{b'c'}{bc}\cos(\beta'-\gamma'-\beta+\gamma).$$

Now $$\beta-\gamma=A-\pi \text{ and } \beta'-\gamma'=A'-\pi,$$
hence $$\beta'-\gamma'-\beta+\gamma=A'-A,$$
and thus we obtain
$$\left(\frac{b'}{b}\right)^2+\left(\frac{c'}{c}\right)^2-2\frac{b'c'}{bc}\cos(A'-A) : \left(\frac{c'}{c}\right)^2+\left(\frac{a'}{a}\right)^2$$
$$-2\frac{c'a'}{ca}\cos(B'-B) : \left(\frac{a'}{a}\right)^2+\left(\frac{b'}{b}\right)^2-2\frac{a'b'}{ab}\cos(C'-C) :: a^2:b^2:c^2.$$

8. Consider next the identity
$$x(y-z)(z-u)(u-y)+z(u-x)(x-y)(y-u)$$
$$=y(z-u)(u-x)(x-z)+u(x-y)(y-z)(z-x).$$

Take any convex pentagon *ABCDE*, and join *AC*, *AD*, *BD*, *BE*, *CE*. Then applying the above identity we obtain
$$(AE).(BC).(CD).(DB)+(CE).(DA).(AB).(BD)$$
$$=(BE).(CD).(DA).(AC)+(DE).(AB).(BC).(CA).$$

Interpreting this, in the usual manner, we obtain

$$\begin{aligned}&AE.BC.CD.DB\cos(EBC+ECD-EDB)\\ &\qquad\qquad+CE.AB.BD.DA\cos(EAB+EBD-EDA)\\ &=BE.AC.CD.DA\cos(EAC+ECD-EDA)\\ &\qquad\qquad+DE.AB.BC.CA\cos(EAB+EBC-ECA);\end{aligned}$$

together with a similar relation in which the cosines of the respective angles are replaced by their sines.

9. Let $ABCDEF$ be a convex hexagon. Join CF, BE, AD, and let P, Q, R, X, Y, Z be the middle points of DE, FA, BC, CF, BE, AD respectively. Let QY and RZ meet in L, RZ and PX in M, and PX and QY in N. Then it is easily proved that

$$(PX)=2\{(EF)-(CD)\},\quad (QY)=2\{(AB)-(EF)\},$$
$$(RZ)=2\{(CD)-(AB)\};$$

consequently, we have

$$(AB).(PX)+(CD).(QY)+(EF).(RZ)=0.$$

Interpreting this, in a similar manner to that in which we did the result in Art. 2, we obtain

$$\frac{AB.PX}{\sin(CDE+DEF+NLM)}=\frac{CD.QY}{\sin(EFA+FAB+LMN)}$$
$$=\frac{EF.RZ}{\sin(ABC+BCD+MNL)}.$$

10. The parabola furnishes us with some very useful methods of interpretation. Thus, consider fig. 4. Draw the tangent to the parabola at some fixed point T, and let the tangents at two other points P and Q meet this tangent in X and Y respectively, and meet each other in R.

Now, the angle TSP is double the angle TSX, and $SX^2=ST.SP$. Hence, if we take ST for our initial line, we have
$$(SX)^2=ST.(SP).$$

Also, since $SX^2=ST.SP$ and $SY^2=ST.SQ$,

we obtain $SX^2.SY^2=ST^2.SP.SQ=ST^2.SR^2$,

and therefore
$$SX.SY=ST.SR.$$

Moreover the angle TSR is equal to the sum of the angles TSX and TSY, hence we have

$$(SX).(SY) = ST.(SR).$$

Thus we have obtained a method of interpreting squares and products.

11. We will utilize the results of the last article to obtain an interpretation of the formula for the square of a polynomial.

Take n fixed points $P_1, P_2, \ldots, P_n$ on the parabola, and draw the tangents at them. Let the tangent at some other point T meet them in $X_1, X_2, \ldots, X_n$. Let Z be the centroid of the n points $X_1, X_2, \ldots, X_n$, O that of the $\frac{1}{2}n(n-1)$ points of intersection of the fixed tangents, and O' that of their points of contact $P_1, P_2, \ldots, P_n$.

Now $\quad n(SZ) = (SX_1) + (SX_2) + \ldots + (SX_n)$;

and, therefore

$$n^2(SZ)^2 = \Sigma(SX)^2 + 2.\Sigma(SX)(SX').$$

But $\quad \Sigma(SX)^2 = n.ST(SO')$

and $\quad 2.\Sigma(SX).(SX') = n(n-1)ST(SO)$,

therefore $\quad n(SZ)^2 = (n-1)ST(SO) + ST(SO')$.

Now, if we join OO' and divide it at P so that, $OP : PO' :: 1 : n-1$, we obtain

$$n.ST(SP) = (n-1)ST(SO) + ST(SO') = n(SZ)^2;$$

and, therefore $\quad (SZ)^2 = ST(SP)$.

Hence, it is evident that P is the point of contact of the tangent through Z. Now P is a fixed point, and therefore the locus of Z is a fixed tangent of the parabola. Thus we obtain the Theorem:—

The locus of the centroid of the points in which a variable tangent to a parabola is intersected by n fixed tangents is a tangent of the parabola, and its point of contact lies on the line joining the centroid of the $\frac{1}{2}n(n-1)$ points of intersection of the fixed tangents to the centroid of their points of contact, and divides that line in the ratio $1 : n-1$.

12. Consider the simplest case of the proposition discussed in the preceding article, namely, when the number of fixed tangents is two. The proposition then reduces to the following:—

The locus of the middle point of the intercept made by two fixed tangents of a parabola upon a variable tangent is the tangent at the extremity of the diameter through the point of intersection of the fixed tangents.

This furnishes us with a very easy method of interpreting formulae including squares or products of binomials. Thus, take the two formulæ

$$(x+y)(z+u) = xz + xu + yz + yu,$$

and

$$x(y'+z') + y(z'+x') + z(x'+y')$$
$$= x'(y+z) + y'(z+x) + z'(x+y).$$

The interpretation of these yields the two following propositions:—

Let $ABCD$ be a convex quadrilateral, and let AD and BC when produced meet in E, and AB and DC when produced meet in F. Draw a parabola touching the four sides of the quadrilateral. Then the tangents to the parabola at the extremities of the diameters through E and F will meet one another at the centroid of the four points A, B, C, D.

PQR and $P'Q'R'$ are two triangles circumscribed to a parabola. Let the tangents at the extremities of the diameters through P', Q', R', P, Q, R respectively meet QR, RP, PQ, $Q'R'$, $R'P'$, $P'Q'$ at the points L, M, N, L', M', N'. Then the triangles LMN and $L'M'N'$ have the same centroid.

13. As an interpretation of the identity

$$(x^2-z^2)(y^2-u^2) = (xy - zu - xu + yz)(xy - zu + xu - yz),$$

we easily deduce the following:—

Tangents are drawn to a parabola at the points P, Q, R, S. The tangents at P and Q meet in K, those at Q and R in L, those at R and S in M, and those at S and P in N. Let A, B, C, D be the middle points of KL, LM, MN, NK; then we have

$$PR \,.\, QS = 4 \,.\, AC \,.\, BD.$$

14. Next take the identity

$$(x^2-yz)(y-z) + (y^2-zx)(z-x) + (z^2-xy)(x-y) = 0.$$

Let PQR be a triangle circumscribed to a parabola; and let QR, RP, PQ touch the parabola at P', Q', R' and meet the tangent at any other point in X, Y, Z. Applying the above identity, we obtain the relation

$$(PP')(YZ) + (QQ')(ZX) + (RR')(XY) = 0.$$

Hence, if O be the point of concurrence of PP', QQ' and RR', it follows that

$$\frac{PP'.YZ}{\sin QOR}=\frac{QQ'.ZX}{\sin ROP}=\frac{RR'.XY}{\sin POQ}.$$

From the same figure we get a very easy interpretation of the identity

$$(y-z)(y^2+yz+z^2)+(z-x)(z^2+zx+x^2)+(x-y)(x^2+xy+y^2)=0.$$

Let L, M, N be the centroid of the triangles $PQ'R'$, $QR'P'$, $RP'Q'$. Then we have

$$(SL).(YZ)+(SM).(ZX)+(SN).(XY)=0,$$

and consequently

$$\frac{SL.YZ}{\sin MSN}=\frac{SM.ZX}{\sin NSL}=\frac{SN.XY}{\sin LSM}.$$

15. We will now obtain an interpretation of the identity

$$\begin{aligned}&yzu(x-y)(x-z)(x-u)+zux(y-z)(y-u)(y-x)\\&\qquad+uxy(z-u)(z-x)(z-y)+xyz(u-x)(u-y)(u-z)\\&=(xy-zu)(xz-yu)(xu-yz);\end{aligned}$$

or as it may be written to suit our purpose of interpretation

$$\begin{aligned}&(xz-yz)(xu-zu)(xy-uy)+(yu-xu)(xy-xz)(yz-ux)\\&\quad+(xz-xu)(yz-yx)(zu-yu)+(yu-yz)(zu-zx)(xu-xy)\\&=(xy-zu)(xz-yu)(xu-yz).\end{aligned}$$

Let $ABCD$ be a convex quadrilateral, and let AB and DC when produced meet in E, and AD and BC when produced meet in F. Now, since a parabola can always be drawn so as to touch four straight lines, we may apply our method to obtain an interpretation of the identity in question from this figure. We obtain

$$\begin{aligned}&(CB).(FA).(DE)+(AD).(BE).(FC)+(AB).(EC).(DF)\\&\qquad+(CD).(BF).(EA)=(FE).(DB).(CA).\end{aligned}$$

Interpreting this in the usual manner, we arrive at the result

$$\begin{aligned}&AB.CE.DF\cos(ABD+CEF+CAF)\\&\qquad+AD.CF.BE\cos(ADB+CFE+CAE)\\&-BC.AF.DE\cos(CFE+ADB+DCA)\\&\quad-CD.AE.BF\cos(CEF+ABD+BCA)=AC.BD.EF.\end{aligned}$$

16. It is easily proved that, if $xx' = yy' = zz'$, then

$$(y-z')(z-x')(x-y') + (y'-z)(z'-x)(x'-y) = 0,$$

and

$$(x-y)(x-y') : (x-z)(x-z') :: (x'-y)(x'-y') : (x'-z)(x'-z').$$

Draw three parabolas having the same focus S. From a point T, external to all three, draw tangents TP and Tp to the first parabola, TQ and Tq to the second, and TR and Tr to the third. Then, taking any line through S for initial line, we have

$$(SP).(Sp) = (SQ).(Sq) = (SR).(Sr) = (ST)^2.$$

Hence, applying the above identities, we obtain

$$(Qr).(Rp).(Pq) + (qR).(rP).(pQ) = 0,$$

and $$(PQ).(Pq) : (PR).(Pr) :: (pQ).(pq) : (pR).(pr),$$

Hence, it follows that

$$Qr.Rp.Pq = qR.rP.pQ,$$

and $$PQ.Pq : PR.Pr :: PQ.pq : pR.pr.$$

17. We can also utilize the parabola to obtain an interpretation of the product of three vectors.

Let PQR be a triangle circumscribed to a parabola (fig. 5); and let QR, RP, PQ meet the tangent at the vertex in X, Y, Z respectively. Draw the circumscribing circle of the triangle PQR; and let O be its centre, and SL the diameter through the focus. Join RL.

Then the angle
SLR = supplement of $SPR = SPY$ = complement of YSP
= complement of $YZP = SZA$;
and, consequently, we have $RSL = ASZ$. Now ASR is equal to the sum of ASX and ASY, and therefore ASL is equal to the sum of ASX, ASY and ASZ. Moreover we have $SZ.SR = AS.SL$, since the triangles ASZ and RSL are similar, and $SX.SY = AS.SR$. Therefore

$$SX.SY.SZ = AS.SR.SZ = AS^2.SL$$

Thus we have

$$(SX).(SY).(SZ) = AS^2.(SL) = 2.AS^2.(SO),$$

taking SA for initial line.

This gives an interpretation for $(SX).(SY).(SZ)$. By making PQZ coincide with RPY we obtain one for $(SX).(SY)^2$; and, by making all three tangents coincide, we obtain one for $(SX)^3$.

18. The results of the last article enable us to obtain an interpretation of the formula for the cube of a binomial.

Let TP and TQ, two tangents to a parabola, meet the tangent at the vertex in X and Y; and let Z be the middle point of XY. Let H and K be the centres of circles passing through the focus S and touching the parabola at P and Q respectively; and let L and M be the centres of the circumscribing circles of the triangles SPT and SQT.

Now

$$2(SZ) = (SX) + (SY),$$

and therefore

$$\begin{aligned} 8.(SZ)^3 &= (SX)^3 + 3.(SX)^2.(SY) + 3.(SX).(SY)^2 + (SY)^3 \\ &= 2.AS^2\{(SH) + 3.(SL) + 3.(SM) + (SK)\} \\ &= 16.AS^2.(SO); \end{aligned}$$

where O is the mean centre of H, K, L, M for multiples m, m, $3m$, $3m$ respectively. Therefore

$$(SZ)^3 = 2.AS^2.(SO).$$

Now the tangent through Z is the tangent at the extremity of the diameter through T; therefore the circle drawn with centre O and radius OS will touch the parabola at the extremity of the diameter through T.

19. Interpreting in like manner the formula

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3(y+z)(z+x)(x+y),$$

we arrive at the following:—

PQR is a triangle circumscribed to a parabola, QR, RP, PQ touching the parabola at P', Q', R' respectively. L, M, N are the centres of circles passing through the focus and touching the parabola at P', Q', R', and H is the centroid of the triangle LMN. K is the centre of the circumscribing circle of the triangle formed by the tangents at the extremities of the diameters through P, Q, R. A point O is taken on HK so that $HO = 8.OK$, and a circle is drawn with centre O so as to pass through the focus. Then this circle will touch the parabola at the point where it is met by the line joining the centroids of the triangles PQR and $P'Q'R'$.

20. In conclusion, we will give interpretations of the identities

$$(x+y+z)(yz+zx+xy) = xyz + (y+z)(z+x)(x+y)$$

and

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-yz-zx-xy).$$

Take the figure of the last article; and let G and G' be the centroids of the triangles PQR and $P'Q'R'$, and U the centroid of the points in which the tangent at the vertex is met by QR, RP, PQ. Also, let F be the centre of circumscribing circle of the triangle PQR. Divide KF at E, so that $FE = 8 . EK$; then we have

$$SU . SG = 2 . AS . SE \text{ and } 2 . AS . HF = 3 . SU . GG'.$$

21. Most of the above results have already been published by me in the mathematical column of the *Educational Times*. A result equivalent to that in Art. 7, was given by Mr. B. Hanumanta Rau in the *Educational Times* for January (No. 8415). I find that in the result of Art. 2 I had been anticipated by Professor Wolstenholme (*Wolstenholme's Mathematical Problems, Nos.* 582 *and* 813).

NOTE. Since writing the above paper I have obtained the following interpretation of the formula

$$x(x-u)(y-z)+y(y-u)(z-x)+z(z-u)(x-y) \\ +(y-z)(z-x)(x-y)=0,$$

by means of the method of Arts. 2, 3, 4, and 5.

Let ABC be a triangle, and P and Q two points within it. Then we have

$$PA . QA . BC \cos(PAB - QAC) + PB . QB . CA \cos(PBC - QBA) \\ + PC . QC . AB \cos(PCA - QCB) = BC . CA . AB.$$

NOTE ON A CLASS OF ALGEBRAICAL IDENTITIES.

By *R. Lachlan, B.A.*, Fellow of Trinity College, Cambridge.

IF x, y, z, w be any quantities such that,

$$x+y+z+w=0,$$

$$ax+by+cz+dw=0,$$

it may be easily proved that

$$Ax^2+By^2+Cz^2+Dw^2=0,$$

$$Aax^2+Bby^2+Ccz^2+Ddw^2=(a^2x+b^2y+c^2z+d^2w)^2,$$

$$A^2x^3+B^2y^3+C^2x^3+D^2w^3=(a^2x+b^2y+c^2z+d^2w)^3,$$

where
$$A=(a-b)(a-c)(a-d),$$

and B, C, D stand for similar expression.

The generalised theorem is:—given n variables x_1, x_2, x_3, ..., x_n satisfying the system of equations

$$\Sigma x=\Sigma ax=\Sigma a^2x=\ldots=\Sigma a^{n-3}x=0,$$

where
$$\Sigma a^p x\equiv a_1{}^p x_1+a_2{}^p x_2+\ldots+a_n{}^p x_n,$$

then
$$\Sigma A^p a^q x^{p+1}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

if $p+q<n-2$;

and
$$\Sigma A^p a^q x^{p+1}=\{\Sigma a^{n-2}x\}^{p+1} \ldots\ldots\ldots\ldots\ldots \text{(II)}$$

if $p+q=n-2$,

where
$$A_r=(a_r-a_1)(a_r-a_2)(a_r-a_3)\ldots(a_r-a_n).$$

To prove this, let

$$\Delta=\begin{vmatrix} 1, & 1, & \ldots, & 1 \\ a_1, & a_2, & \ldots, & a_{n-1} \\ a_1{}^2, & a_2{}^2, & \ldots, & a^2_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_1{}^{n-2}, & a_2{}^{n-2}, & \ldots, & a_{n-1}^{n-2} \end{vmatrix},$$

and let $\mu_1, \mu_2, \mu_3, \ldots, \mu_{n-1}$ denote the minors of the constituents of the last row. Also, let Δ', μ'_1, μ'_2, &c. denote what these become when a_n is substituted for a_{n-1}.

We have then

$$(a_r - a_n)\,\Delta = A_r\mu_r,$$

$$\mu_{n-1} = \mu'_{n-1} = \mu \text{ (say).}$$

Now, solving the given equations so as to obtain each variable x in terms of x_{n-1} and x_n, we find

$$\mu x_r = \mu_r x_{n-1} + \mu_r' x_n = \frac{\Delta}{A_r}(a_r - a_n)\,x_{n-1} + \frac{\Delta'}{A_r}(a_r - a_{n-1})\,x_n.$$

Hence

$$\mu^{p+1}\Sigma A^p a^q x^{p+1} = \sum_1^{n-2}{}_r \frac{a^q{}_r}{A_r}\left\{\Delta\,(a_r - a_n)\,x_{n-1} + \Delta'\,(a^r - a_{n-1})\,x_n\right\}^{p+1}$$

$$+ \mu a^q{}_{n-1}\,\Delta^p\,(a_{n-1} - a_n)^p x_{n-1}^{p+1}$$

$$+ \mu\,a_n{}^q\Delta'^p\,(a_n - a_{n-1})^p x_n{}^{p+1}.$$

In this expression the coefficient of x_{n-1}^{p+1}

$$= \Delta^p\,\{\sum_1^{n-2}{}_r a_r{}^q\mu_r\,(a_r - a_n)^p + \mu a^q{}_{n-1}\,(a_{n-1} - a_n)^p\},$$

which clearly vanishes if $q + p < n - 2$ but if $q + p = n + 2$, the value of the expression is Δ^{p+1}.

Again, the coefficient of $x_{n-1}^{t+1}x_n{}^{p-t}$

$$= \sum_1^{n-2}{}_r a_r{}^q\mu_r\,(a_r - a_n)^t\,(a_r - a_{n-1})^{p-t}\Delta^t\Delta'^{p-t}\frac{(p+1)\,!}{(t+1)\,!\,(p-t)\,!},$$

this also clearly vanishes if $p + q < n - 2$, and is equal to

$$\frac{(t+1)\,!\,(p-t)\,!}{(p+1)\,!}\,\Delta^{t+1}\Delta'^{p-t}$$

when $p + q = n - 2$.

Hence, if $p + q < n - 2$,

$$\Sigma A^p a^q x^{p+1} = 0,$$

and, if $p + q = n - 2$,

$$\mu^{p+1}\Sigma A^p a^q x^{p+1} = (\Delta x_{n-1} + \Delta' x_n)^{p+1},$$

and taking $p = 0$, $q = n - 2$, we have

$$\mu\Sigma a^{n-2}x = \Delta x_{n-1} + \Delta' x_n\,;$$

therefore $\quad \Sigma A^p a^q x^{p+1} = (a_1{}^{n-2}x_1 + a_2{}^{n-2}x^2 + \ldots + a_n{}^{n-2}x_n)^{p+1}.$

A PROOF OF FOURIER'S SERIES FOR PERIODIC FUNCTIONS.

By *Peter Alexander*.

Let $[\phi(x)]^p$ denote

$$\frac{1}{2l}\int_q^p f(v)\,dv+\frac{1}{l}\,\Sigma_{m=1}^{m=\infty}\int_q^p f(v)\,.\cos\frac{m\pi(v-x)}{l}\,dv,$$

where $p-q$ is not greater than $2l$.

Then $[\phi(x)]^p_q$

$$=\frac{1}{l}\int_q^p\left\{\tfrac{1}{2}+\cos\frac{\pi(v-x)}{l}+\cos\frac{2\pi(v-x)}{l}+\&c.\right\}f(v)\,dv$$

$$=\frac{1}{l}\,L_{h=1}\int_q^p\left\{\tfrac{1}{2}+h\cos\frac{\pi(v-x)}{l}+h^2\cos\frac{2\pi(v-x)}{l}+\&c.\right\}f(v)\,dv$$

$$=\frac{1}{2l}L_{h=1}\int_q^p\frac{1-h^2}{1-2h\cos\frac{\pi(v-x)}{l}+h^2}\,.f(v)\,dv$$

$$=\frac{1}{2l}\,L_{h=1}\int_q^p\frac{\frac{1-h}{1+h}\sec^2\frac{\pi(v-x)}{2l}}{\left(\frac{1-h}{1+h}\right)^2+\tan^2\frac{\pi(v-x)}{2l}}\,.f(v)\,dv$$

$$=\frac{1}{\pi l}L_{a=0}\int_\beta^\alpha\frac{a}{a^2+z^2}f\left(x+\frac{2l}{\pi}\tan^{-1}z\right)dz\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)},$$

where α and β denote $\tan\frac{\pi(p-x)}{2l}$ and $\tan\frac{\pi(q-x)}{2l}$ respectively.

From this it is apparent that unless z become zero somewhere on or between the limits of integration $[\phi(x)]$ will be zero because all its elements are zero.

That is $$[\phi(x)]_q^p = 0,$$

$$\left.\begin{array}{ll}\text{if} & (i+1)\pi > \dfrac{\pi(p-x)}{2l} > \dfrac{\pi(q-x)}{2l} > i\pi, \\ \text{or} & (i+1)\pi > \dfrac{\pi(x-q)}{2l} > \dfrac{\pi(x-p)}{2l} > i\pi, \\ \text{that is if} & q-2il > x > p-2(i+1)l, \\ \text{or} & q+2(i+1)l > x > p+2il,\end{array}\right\}\text{......(II)},$$

where i is a positive integer or zero.

Taking $i=0$ which is the case with which we are concerned at present,

$$\left.\begin{array}{ll} & [\phi(x)]_q^p = 0, \\ \text{if} & q > x > p-2l, \\ \text{or} & q+2l > x > p\end{array}\right\}\text{..................(II).*}$$

Again if $x=p$ and $p-q<2l$, then from (I)

$$[\phi(x)]_q^p = [\phi(x)]_q^x$$

$$= \frac{1}{2l} L_{a=0} \int_\beta^0 \frac{a}{a^2+z^2} f\left(x+\frac{2l}{\pi}\tan^{-1}z\right) dz$$

$$= \frac{1}{2l} L_{a=0} \left\{\int_{-\sqrt{a}}^0 + \int_{-\gamma}^{-\sqrt{\alpha}}\right\} \frac{a}{a^2+z^2} f\left(x+\frac{2l}{\pi}\tan^{-1}z\right) dz.$$

where β and γ denote $\tan\dfrac{\pi(q-x)}{l}$ and $\tan\dfrac{\pi(x-q)}{2l}$ respectively.

Now the second term of this is zero by (II)*; therefore

$$[\phi(x)]_q^{p=x} = \frac{1}{2l} L_{a=0} \left\{\int_{-\sqrt{a}}^0 \frac{a}{a^2+z^2} \cdot f\left(x+\frac{2l}{\pi}\tan^{-1}z\right) dz \text{...(III)},\right.$$

Hence if z_1 and z_2 be the values of z, for which

$$f\left(x+\frac{2l}{\pi}\tan^{-1}z\right)$$

has respectively its greatest and least values from $z=+\sqrt{a}$ to $z=0$ both inclusive.

Then

$$\int_{-\sqrt{a}}^{0} \frac{a}{a^2+z^2} f\left(x^2 + \frac{2l}{\pi}\tan^{-1} z\right) dz$$

$$< f\left(x + \frac{2l}{\pi}\tan^{+1} z_1\right) \int_{-\sqrt{a}}^{0} \frac{a\,dz}{a^2+z^2}$$

$$< f\left(x + \frac{2l}{\pi}\tan^{-1} z_1\right) \tan^{-1}\left(\frac{1}{\sqrt{a}}\right)$$

and

$$< f\left(x + \frac{2l}{\pi}\tan^{-1} z_2\right) \tan^{-1}\left(\frac{1}{\sqrt{a}}\right).$$

But the limit of both of these, when a is indefinitely diminished, is

$$f(x).\tan^{-1}(\infty) \text{ or } \frac{\pi}{2} f(x).$$

Therefore

$$L_{a=0} \int_{-\sqrt{a}}^{0} \frac{a}{a^2+z^2} f\left(x + \frac{2l}{\pi}\tan^{-1} z\right) dz = \frac{\pi}{2} f(x).$$

Hence (III) becomes

$$\left.\begin{array}{l} [\phi(x)]_q^{p=x} = \tfrac{1}{2} f(x) \\ \text{where} \quad p - q < 2l \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(\text{IV}).$$

Similarly it may be shown that

$$\left.\begin{array}{l} [\phi(x)]_{q=x}^{p} = \tfrac{1}{2} f(x) \\ \text{where} \quad p - q < 2l \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(\text{V}).$$

Again, if $p > x > q$,

$$\left.\begin{array}{l} [\phi(x)]_q^p = [\phi(x)]_x^p + [\phi(x)]_q^x \\ \qquad = \tfrac{1}{2} f(x+0) + \tfrac{1}{2} f(x-0) \text{ by (V) and (IV)} \\ \qquad = \dfrac{f(x+0) + f(x-0)}{2} \end{array}\right\}\ldots\ldots(\text{VI}).$$

If there be no discontinuity at x (VI) will become

$$\left.\begin{array}{l} [\phi(x)]_q^p = f(x) \\ \text{where} \quad p > x > q \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(\text{VI})^*.$$

Again, since

$$\cos\frac{m\pi(v-x)}{l} = \cos\frac{m\pi\{v-(x+2l)\}}{l},$$

$$\left.\begin{aligned} &\text{therefore} & [\phi(x)]_q^{x+2l} &= [\phi(x+2l)]_q^{x+2l} \\ & & &= \tfrac{1}{2} f(x+2l) \text{ by (IV.)} \\ &\text{if } x < q, \text{ and therefore} & (x+2l) - q &< 2l \end{aligned}\right\} \text{.......(VII).}$$

$$\left.\begin{aligned} [\phi(x)]_x^{x+2l} &= [\phi(x)]_x^q + [\phi(x)]_q^{x+2l} \\ &= \tfrac{1}{2} f(x) + \tfrac{1}{2} f(x+2l) \text{ by (V) and (VII)} \\ &= \frac{f(x) + f(x+2l)}{2} \\ \text{where} \quad & x + 2l > q > x \end{aligned}\right\} \text{..(VIII).}$$

Similarly it may be shewn that

$$[\phi(x)]_{x-2l}^{x} = \frac{f(x) + f(x+2l)}{2} \text{............(IX).}$$

Wherefore from (VI),* (IV) and (V), (II)* we conclude that

$$\left.\begin{aligned} \text{if} \qquad & p - q < 2l \\ & [\phi(x)]_q^p = f(x),\ \tfrac{1}{2} f(x), \text{ or } 0 \end{aligned}\right\} \text{............(X).}$$

according as x lies within, upon or without the limits p and q, but within the limits $q + 2l$ and $p - 2l$, unless $f(v)$ is discontinuous at the point $v = x$ and $p > x > q$, in which case

$$[\phi(x)]_q^p = \frac{f(x+0) + f(x-0)}{2},$$

Also from (VI),* (VIII), and (IX) it follows that

$$\left.\begin{aligned} \text{if} \qquad & p - q = 2l \\ & [\phi(x)]_q^p = f(x), \text{ or } \frac{f(p) + f(q)}{2} \end{aligned}\right\} \text{.........(X),*}$$

according as x lies within or upon the limits p and q.

But if x is a point of discontinuity between p and q

$$[\phi(x)]_q^p = \frac{f(x+0) + f(x-0)}{2}.$$

Equations (X) and (X)* are Fourier's Series-Theorem.

Throughout it has been assumed that $f(x)$ is nowhere infinite.

ON CAYLEY'S DIFFERENTIAL EQUATION FOR ORTHOGONAL SURFACES.

By *A. R. Johnson.*

THE following article deduces Cayley's Equation as a particular case of the differential equation of surfaces whose principal tangents twist round at a given rate as the point of contact travels along a trajectory, and calls attention to the form of the coefficients, shewing that they all vanish at an umbilic. In order to make the article more complete, a short account is given of Catalan's method of finding the other two equations of a triple system when one of them is given.

1. Let l, m, n be the direction cosines of the normal at a point on one of a system of surfaces, $\partial\nu$ the element of normal intercepted between this surface and the consecutive one, and let $\partial\theta$ be the angle between the planes through the normal element and the corresponding principal tangents at its extremities to the two surfaces. Let $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ be the direction cosines of the two principal tangents on the surface, then $\alpha_1+\alpha_2\partial\theta$, $\beta_1+\beta_2\partial\theta_1$, $\gamma_1+\gamma_2\partial\theta$; $\alpha_2-\alpha_1\partial\theta$, $\beta_2-\beta_2\partial\theta$, $\gamma_2-\gamma_1\partial\theta$; are the direction cosines of the principal tangents on the consecutive.

The equations giving the curvature and the direction cosines of the principal tangents at any point of the surface are

$$\left.\begin{array}{llll}\alpha\left(\dfrac{dl}{dx}-\dfrac{1}{\rho}\right) & +\beta\,\dfrac{dl}{dy} & +\gamma\,\dfrac{dl}{dz} & =0\\ \alpha\,\dfrac{dm}{dx} & +\beta\left(\dfrac{dm}{dy}-\dfrac{1}{\rho}\right) & +\gamma\,\dfrac{dm}{dz} & =0\\ \alpha\,\dfrac{dn}{dy} & +\beta\,\dfrac{dn}{dy} & +\gamma\left(\dfrac{dn}{dz}-\dfrac{1}{\rho}\right) & =0\end{array}\right\}\ldots\ldots(1)$$

and these relations must be satisfied when for α, β, γ; we substitute $\alpha_1+\alpha_2\partial\theta$, $\beta_1+\beta_2\beta\partial$, $\gamma_1+\gamma_2\partial\theta$, and at the same time give to the other quantities their values at the other end of the normal element.

We have then, effecting these changes, subtracting the corresponding equations in (1) with suffix (1) attached to α, β, γ, ρ, and dividing by $\partial\nu$

$$\left\{\alpha_2\left(\frac{dl}{dx}-\frac{1}{\rho}\right)+\beta_2\frac{dl}{dy}+\gamma_2\frac{dl}{dz}\right\}\frac{d\theta}{d\nu}+\alpha_1\frac{d}{d\nu}\left(\frac{dl}{dx}-\frac{1}{\rho_1}\right)$$
$$+\beta_1\frac{d}{d\nu}\frac{dl}{dy}+\gamma_1\frac{d}{d\nu}\frac{dl}{dz}=0$$

and two similar equations, or, since

$$\alpha_2\frac{dl}{dx}+\beta_2\frac{dl}{dy}+\gamma_2\frac{dl}{dz}=\frac{\alpha_2}{\rho_2},\ \text{\&c.},$$

$$\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\alpha_2\frac{\partial\theta}{\partial\nu}=\alpha_1\frac{d}{d\nu}\left(\frac{dl}{dx}-\frac{1}{\rho_1}\right)+\beta_1\frac{d}{d\nu}\frac{dl}{dy}+\gamma_1\frac{d}{d\nu}\frac{dl}{dz},$$

$$\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\beta_2\frac{\partial\theta}{\partial\nu}=\alpha_1\frac{d}{d\nu}\frac{dm}{dx}+\beta_1\frac{d}{d\nu}\left(\frac{dm}{dy}-\frac{1}{\rho_1}\right)+\gamma_1\frac{d}{d\nu}\frac{dm}{dz},$$

$$\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\gamma_2\frac{\partial\theta}{d\nu}=\alpha_1\frac{d}{d\nu}\frac{dn}{dx}+\beta_1\frac{d}{d\nu}\frac{dn}{dy}+\gamma_1\frac{d}{d\nu}\left(\frac{dn}{dz}-\frac{1}{\rho_1}\right).$$

Multiply these three equations by α_2, β_2, γ_2 and add, then

$$\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\frac{\partial\theta}{d\nu}=\alpha_1\alpha_2\frac{d}{d\nu}\frac{dl}{dx}+\beta_1\beta_2\frac{d}{d\nu}\frac{dm}{dy}+\gamma_1\gamma_2\frac{d}{d\nu}\frac{dm}{dz}$$

$$+\beta_1\gamma_2\frac{d}{d\nu}\frac{dn}{dy}+\beta_2\gamma_1\frac{d}{d\nu}\frac{dm}{dz}+\gamma_1\alpha_2\frac{d}{d\nu}\frac{dl}{dz}+\gamma_2\alpha_1\frac{d}{d\nu}\frac{dn}{dx}$$

$$+\alpha_1\beta_2\frac{d}{d\nu}\frac{dm}{dz}+\alpha_2\beta_1\frac{d}{d\nu}\frac{dl}{dy},$$

and altering the sign of $\partial\theta$ and interchanging the suffixes there results a similar equation which, added to this, gives

$$\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\frac{\partial\theta}{\partial\nu}=2\alpha_1\alpha_2\frac{d\nu}{d}\frac{dx}{dl}+2\beta_1\beta_2\frac{d\nu}{d}\frac{dm}{dy}+2\gamma_1\gamma_2\frac{dn}{dz}$$

$$+(\beta_1\gamma_2+\beta_2\gamma_1)\frac{d}{d\nu}\left(\frac{dm}{dz}+\frac{dn}{dy}\right)+(\gamma_1\alpha_2+\gamma_2\alpha_1)\frac{d}{d\nu}\left(\frac{dn}{dx}+\frac{dl}{dz}\right)$$

$$+(\alpha_1\beta_2+\alpha_2\beta_1)\frac{d}{d\nu}\left(\frac{dl}{dy}+\frac{dm}{dz}\right)\ \ldots\ldots\ldots\ldots\ (2).$$

2. The direction cosines $\alpha\beta\gamma$ are given by

$$l\alpha+m\beta+n\gamma=0,$$

and

$$\alpha\frac{dl}{dx}+\beta\frac{dm}{dy}+\gamma\frac{dl}{dz}:\alpha=\alpha\frac{dm}{dx}+\beta\frac{dm}{dy}+\gamma\frac{dm}{dz}:\beta$$

$$=\alpha\frac{dn}{dx}+\beta\frac{dn}{dy}+\gamma\frac{dn}{dz}\gamma,$$

whence results the relation

$$\beta^2\left(l\frac{dn}{dy}-m\frac{dn}{dx}\right)+\gamma^2\left(n\frac{dm}{dx}-l\frac{dm}{dz}\right)$$
$$-\beta\gamma\left\{l\left(\frac{dm}{dy}-\frac{dn}{dz}\right)-m\frac{dm}{dx}+n\frac{dn}{dx}\right\}=0.$$

From this and the two analagous equations, there results

$$\alpha_1\alpha_2 : m\frac{dl}{dz}-m\frac{dl}{dy}=\beta_1\beta_2 : n\frac{dm}{dx}-l\frac{dm}{dz}=\gamma_1\gamma_2 : l\frac{dn}{dy}-m\frac{dn}{dx}$$
$$=\beta_1\gamma_2+\beta_2\gamma_1 : l\left(\frac{dm}{dy}-\frac{dn}{dz}\right)-m\frac{dm}{dx}+n\frac{dn}{dx}$$
$$=\gamma_1\alpha_1+\gamma_2\alpha_1 : m\left(\frac{dn}{dx}-\frac{dl}{dx}\right)-n\frac{dn}{dy}+l\frac{dl}{dy}$$
$$=\alpha_1\beta_2+\alpha_2\beta_1 : n\left(\frac{dl}{dx}-\frac{dm}{dy}\right)-l\frac{dl}{dz}+m\frac{dm}{dz}=\frac{1}{H},\ \text{say},$$

where

$$H^2=\left\{l\left(\frac{dm}{dy}-\frac{dn}{dz}\right)-m\frac{dm}{dx}+n\frac{dn}{dx}\right\}\ +\ldots$$
$$-4\left(l\frac{dn}{dy}-m\frac{dn}{dx}\right)\left(n\frac{dm}{dx}-l\frac{dm}{dz}\right)-\ldots\quad\ldots\ldots(3).$$

Hence substituting in (3), there results

$$\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)^2\frac{\partial\theta}{\partial\nu}=2\left(m\frac{dl}{dz}-n\frac{dl}{dy}\right)\frac{d}{d\nu}\left(\frac{dl}{dx}\right)$$
$$+2\left(n\frac{dm}{dx}-l\frac{dm}{dz}\right)\frac{d}{d\nu}\left(\frac{dm}{dy}\right)+2\left(l\frac{dn}{dy}-m\frac{dn}{dx}\right)\frac{d}{d\nu}\left(\frac{dn}{dz}\right)$$
$$+\left\{l\left(\frac{dm}{dy}-\frac{dn}{dz}\right)-m\frac{dm}{dx}+n\frac{dn}{dx}\right\}\frac{d}{d\nu}\left(\frac{dm}{dz}+\frac{dn}{dy}\right)$$
$$+\left\{m\left(\frac{dn}{dz}-\frac{dl}{dx}\right)-n\frac{dn}{dy}+l\frac{dl}{dy}\right\}\frac{d}{d\nu}\left(\frac{dn}{dx}+\frac{dl}{dz}\right)$$
$$+\left\{n\left(\frac{dl}{dx}-\frac{dm}{dy}\right)-l\frac{dl}{dz}+m\frac{dm}{dz}\right\}\frac{d}{d\nu}\left(\frac{dl}{dy}+\frac{dm}{dx}\right)\ldots\ldots(A);$$

replacing H by its value $\frac{1}{\rho_1} - \frac{1}{\rho_2}$, a relation proved below, (A) is the differential equation sought. By equating $\frac{\partial\theta}{d\nu}$ to zero we get, as is shewn by Levy's Theorem, the equation to all orthogonal surfaces.

3. To reduce (A) to Cayley's form, refer to the normal and principal tangents, and let the equation to the surface neglecting powers of coordinates above the third become

$$0 = z + u + v,$$

where u and v are of the second and third degree in the coordinates, and $u_{11} = -\frac{1}{\rho_1}$, $u_{22} = -\frac{1}{\rho_2}$, $u_{12} = 0$.

Hence
$$\left.\begin{aligned} l &= u_1(1 - u_3) + v_1 \\ m &= u_2(1 - u_3) + v_2 \\ n &= 1 - \tfrac{1}{2}(u_1^2 + u_2^2) + v_3 \end{aligned}\right\}.$$

Whence, substituting in (A) and making x, y, z all zero,

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{\partial\theta}{d\nu} = 2\,(v_{123} - 2u_{13}\,.\,u_{23})\ldots\ldots\ldots\ldots(4),$$

or, since $\mathfrak{h}^2 = u_1^2 + u_2^2 + u_3^2 = 1$ at the origin, we have on the introduction of h to make up the proper dimensions

$$\frac{\mathfrak{h}^2}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{\partial\theta}{\partial\nu} = \mathfrak{h}\,.\,v_{123} - 2u_{13}\,.\,u_{23}\ (xyz \text{ all zero})$$

$$= \mathfrak{h}\,.\,F_{123} - 2F_{13}\,.\,F_{23}\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

if $F = r$ be the equation to the surface, 1, 2, 3, denoting fluxes along principal tungents and normal.
So that

$$F_{123} = \frac{d}{d\nu}\left\{\alpha_1\frac{d}{dx} + \beta_1\frac{d}{dy} + \gamma_1\frac{d}{dz}\right\}\left\{\alpha_2\frac{d}{dx} + \beta_2\frac{d}{dy} + \gamma_2\frac{d}{dz}\right\}F$$

$$= \left\{\mathfrak{A}\frac{da}{d\nu} + \mathfrak{B}\frac{db}{d\nu} + \mathfrak{C}\frac{dc}{d\nu} + 2\mathfrak{F}\frac{df}{d\nu}\right.$$

$$\left. + 2\,\mathfrak{G}\frac{dg}{d\nu} + 2\mathfrak{H}\frac{dh}{d\nu}\right\} \div \mathfrak{h}^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right),$$

where $$\mathfrak{A} = \mathfrak{h}^2 \left\{ m \frac{dl}{dz} - n \frac{dl}{dy} \right\}, \text{ etc.}$$

$$\mathfrak{F} = \mathfrak{h}^2 \left\{ l \left(\frac{dm}{dy} - \frac{dn}{dz} \right) - m \frac{dm}{dx} + n \frac{dn}{dx} \right\}, \text{ etc.,}$$

which may be easily shewn to be identical with the values given in *Salmon*, p. 448, 4th ed.

Also

$$F_{13} . F_{23} = \frac{d}{d\nu} \left(\alpha_1 \frac{d}{dx} + \beta_1 \frac{d}{dy} + \gamma_1 \frac{d}{dz} \right) F$$

$$\times \frac{d}{d\nu} \left(\alpha_2 \frac{d}{dx} + \beta_2 \frac{d}{dy} + \gamma_1 \frac{d}{dz} \right) F$$

$$= \left(\alpha_1 \frac{dL}{d\nu} + \beta_1 \frac{dM}{d\nu} + \gamma_1 \frac{dN}{d\nu} \right) \left(\alpha_2 \frac{dL}{d\nu} + \beta_2 \frac{dM}{d\nu} + \gamma_2 \frac{dN}{d\nu} \right)$$

$$= (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}) \left(\frac{dL}{d\nu}, \frac{dM}{d\nu}, \frac{dN}{d\nu} \right)^2$$

$$\div \mathfrak{h}^4 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right).$$

Hence substituting in (5) and putting

$$F = \mathfrak{h} \frac{d}{d\nu} = L \frac{d}{dx} + M \frac{d}{dy} + N \frac{d}{dz}$$

there results

$$\frac{\mathfrak{h}^8}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)^2 \frac{\partial \theta}{\partial \nu} = \mathfrak{h}^2 \{ \mathfrak{A} \delta a + \mathfrak{B} \delta b + \mathfrak{C} \delta c + 2 \mathfrak{F} \delta f + 2 \mathfrak{G} \delta g$$

$$+ 2 \mathfrak{H} \delta h \} - 2 (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\delta L, \delta M, \delta N)^2 \ldots\ldots\ldots (B).$$

which on putting $\frac{\partial \theta}{\partial \nu} = 0$, gives Cayley's differential equation affected by the redundant factor h^2. Using the values of l, m, n above the values of H^2, we easily verify that

$$H^2 = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)^2.$$

4. Equation (B) shews that the right-hand member vanishes at every umbilic on any surface. In fact, if we put

$$bN^2 + cM^2 - 2fMN = LMN . P,$$
$$cL^2 + aN^2 - 2gNL = LMN . Q,$$
$$aM^2 + bL^2 - 2hLM = LMN . R.$$

We have have, as may be easily shewn,

$$\left.\begin{array}{l}\mathfrak{A} = L^2P(M^2 - N^2) + \{M^2R - N^2Q\}(M^2 + N^2)\\ \mathfrak{B} = M^2Q(N^2 - L^2) + \{N^2P - L^2R\}(N^2 + L^2)\\ \mathfrak{C} = N^2R(L^2 - M^2) + \{L^2Q - M^2P\}\{L^2 + M^2)\\ \mathfrak{F} = MN\{(L^2 + N^2)R - (L^2 + M^2)Q\}\\ \mathfrak{G} = NL\{(M^2 + L^2)P - (M^2 + N^2)R\}\\ \mathfrak{H} = LM\{(N^2 + M^2)Q - (N^2 + L^2)P\}\end{array}\right\}\ldots(C).$$

so that the coefficients are all expressible in terms of P, Q, R and the first differential coefficients; and each coefficient vanishes when

$$\frac{P}{M^2 + N^2} = \frac{Q}{N^2 + L^2} = \frac{R}{L^2 + M^2}$$

the conditions for an umbilic.

5. Catalan's method (*Comptes Rendus*, LXXIX) may be set forth in a few lines.

Let $u = c$, $v = c'$ be two independent integrals of the equations

$$dx : \frac{df}{dx} = dy : \frac{df}{dy} = dz : \frac{df}{dz} \ldots\ldots\ldots\ldots\ldots (1)',$$

$f(x, y, z, r) = 0$ being a solution of Cayley's Equation, and the variable parameter r eliminated from $(1)'$. Then the two orthogonal systems being each a collection of trajectories must be given by equations of the form

$$\phi(u, v) = 0, \quad \psi(u, v) = 0;$$

and we must have in addition

$$\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 = 0,$$

or

$$\frac{d\phi}{du}\frac{d\psi}{du}(u_1^2 + u_2^2 + u_3^2) + \left(\frac{d\phi}{du}\frac{d\psi}{dv} + \frac{d\phi}{du}\frac{d\psi}{dv}\right)(u_1v_1 + u_2v_2 + u_3v)$$

$$+ \frac{d\phi}{du}\frac{d\psi}{dv}(v_1^2 + v_2^2 + v_3^2) = 0.$$

Put

$$\left.\begin{array}{l}\dfrac{d\phi}{du} + A\dfrac{d\phi}{dv} = 0\\ \dfrac{d\psi}{du} + B\dfrac{d\phi}{dv} = 0\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (3)',$$

where A and B must be functions of u and v, then substituting in $(2)'$,

$$AB(u_1^2+u_2^2+u_3^2)-(A+B)(u_1v_1+u_2v_2+u_3v_3)+v_1^2+v_2^2+v_3^2=0.$$

If then the system be an orthogonal one, it must be possible to find functions $g(u, v)$, $h(u, v)$ of u and v, such as to make

$$g(u, v)(u_1^2+u_2^2+u_3^2)-h(u, v)(u_1v_1+u_2v_2+u_3v_3)+v_1^2+v_2^2+v_3^2=0,$$

and having found these functions, A and B are to be determined as the roots of the equation

$$A^2-Ah(u, v)+g(u, v)=0,$$

the results substituted in $(3)'$ and then $(3')$ solved by Lagrange's method.

NOTE ON THE CYCLIDE.

By *A. R. Johnson.*

THE five-fold generation of the cyclide can be deduced very simply by identifying the equation referred to the principal spheres, with that referred to the principal planes, and at the same time simple expressions can be obtained for the elements of the guiding quadric and the Jacobian sphere.

For, denoting by a, b, c the semi-axes of quadric, and by α, β, γ, ρ the coordinates of the centre and the radius of Jacobian, the equation referred to the principal planes of the ellipsoid is

$$\{x^2+y^2+z^2-(\alpha^2+\beta^2+\gamma^2-\rho^2)\}^2$$
$$-4\{a^2(x-\alpha)^2+b^2(y-\beta)^2+c^2(z-\gamma)^2\}=0;$$

and if x_1, y_1, z_1, r_1, &c. be the elements of the principal spheres, this equation must be identical with

$$\Sigma A_1(x^2+y^2+z^2+x_1^2+y_1^2+z_1^2-r_1^2)^2$$
$$-4\Sigma A_1(x^2+y^2+z^2+x_1^2+y_1^2+z_1^2-r_1^2)(xx_1+yy_1+zz_1)$$
$$+4\Sigma A\ (xx_1+yy_1+zz_1)^2=0.$$

On comparison, there result the relations

$$\left.\begin{array}{l}\Sigma m_1 x_1 = \Sigma m_1 y_1 = \Sigma m z_1 = 0\\ \Sigma m_1 y_1 z_1 = \Sigma m_1 z_1 x_1 = \Sigma m x_1 y_1 = 0\end{array}\right\}\text{..........}(1),$$

$$\left.\begin{array}{l}2\Sigma m_1(x_1^2+y_1^2+z_1^2-r_1^2)+4\Sigma m_1 x_1^2=-2(\alpha^2+\beta^2+\gamma^2-\rho^2)-4a^2\\ 2\Sigma m_1(x_1^2+y_1^2+z_1^2-r_1^2)+4\Sigma m_1 y_1^2=-2(\alpha^2+\beta^2+\gamma^2-\rho^2)-4b^2\\ 2\Sigma m_1(x_1^2+y_1^2+z_1^2-r_1^2)+4\Sigma m_1 z_1^2=-2(\alpha^2+\beta^2+\gamma^2-\rho^2)-4c^2\end{array}\right\}\text{...}(2),$$

$$\left.\begin{array}{l}8L \equiv -4\Sigma m_1(x_1^2+y_1^2+z_1^2-r_1^2)\,x_1 = 8a^2\alpha\\ 8M \equiv -4\Sigma m_1(x_1^2+y_1^2+z_1^2-r_1^2)\,y_1 = 8b^2\beta\\ 8N \equiv -4\Sigma m_1(x_1^2+y_1^2+z_1^2-r_1^2)\,z_1 = 8c^2\gamma\end{array}\right\}\text{.....}(3),$$

$$\begin{aligned}Q &\equiv \Sigma m_1(x_1^2+y_1^2+z_1^2-r_1^2)^2\\ &= (\alpha^2+\beta^2+\gamma^2-\rho^2)^2-4(a^2\alpha^2+b^2\beta^2+b^2\gamma^2)\text{...}(4),\end{aligned}$$

m_1, m_2, m_3, m_4, m_5 denoting the quotients of A_1, A_2, A_3, A_4, A_5 by ΣA.

From (1) is deduced at once:—

Load the centres of the five principal spheres with the masses m_1, m_2, m_3, m_4, m_5. *The principal planes of the guiding quadric are those of the quintad of mass points.*

From (2),

$$a^2 = I_1 - K,\quad b^2 = I_2 - K,\quad c^2 = I_3 - K,$$

where I_1, I_2, I_3 are the moments of inertia of the quintad about its principal axes; hence:—*The guiding quadric is confocal with the ellipsoid of gyration of the quintad.*

Also $$\alpha^2+\beta^2+\gamma^2-\rho^2 = 2K + P^2 - 3R^2\text{.........}(5),$$

where R is the radius of gyration of the quintad and

$$P^2 = \Sigma m_1 r_1^2.$$

From (3),

$$\alpha = \frac{L}{I_1-K},\quad \beta = \frac{M}{I_2-K},\quad \gamma = \frac{N}{I_3-K}\text{........}(6),$$

and so, from (5),

$$\rho^2 = 3R^2 - P^2 - 2K + \left\{\left(\frac{L}{I_1-K}\right)^2 + \left(\frac{M}{I_2-K}\right)^2 + \left(\frac{N}{I_3-K}\right)^2\right\},$$

giving the coordinates of the centre and the radius of the Jacobian.

Lastly, substituting from (5) and (6) in (4), there results

$$Q = (2K + P^2 - 3R^2)^2 - 4\left\{\frac{L^2}{I_1-K} + \frac{M^2}{I_2-K} + \frac{N^2}{I_3-K}\right\}\text{...}(7),$$

a quintic equation in K, proving the five-foldness of generation.

It is to be remarked that the spheres need not be the principal spheres in the above proof, and that their number may be any whatever, and with obvious modifications the above results are still true. And, as the equation to the cyclide can be reduced in an infinity of ways to the form

$$aX^2+bY^2+cZ^2+dV^2=0,$$

X, Y, Z, W being four spheres, the above method proves the five-fold generation directly from the definition of the cyclide.

From the form of (7), it is evident that in space of n dimensions the generation of the quasi-cyclide is $(n+2)$-fold. This includes as a particular case the four-fold generation of the bicircular quartic, whence follows at once the proof of the existence of the sixteen foci arranged in fours on four circles.

NOTE ON THEOREMS IN WEIERSTRASS'S THEORY OF ELLIPTIC FUNCTIONS.

By *Arthur Buchheim, M.A.*

THE following proof of the equation $\eta\omega'-\eta'\omega=\pm\pi i$ is not presented as being simpler than that given in Mr. Forsyth's *Note on Weierstrass's Theory of Doubly Periodic Functions*, but only as perhaps connecting itself more naturally with the formulæ which in Schwarz's *Formeln und Lehrsätze* immediately precede the enunciation of the theorem.

We have by (5) on p. 8 of the *Formeln und Lehrsätze*

$$\sigma(u)=\exp\frac{\eta u^2}{2\omega}\cdot\frac{2\omega}{\pi}\sin\frac{u\pi}{2\omega}\prod_1^\infty a\left(1-\frac{\sin^2\dfrac{u\pi}{2\omega}}{\sin^2\dfrac{n\omega\pi}{\omega}}\right).$$

It follows from this that $\sigma(u+2\omega)=-\exp 2\eta(u+\omega)\,\sigma(u)$, and then that $\eta=\sigma'(\omega)/\sigma(\omega)$; η' is defined as $\sigma'(\omega')/\sigma(\omega')$.

We easily get

$$\frac{\sigma' u}{\sigma u}=\frac{\eta u}{\omega}+\frac{\pi}{2\omega}\cot\frac{\pi u}{2\omega}-\frac{\pi}{2\omega}\sum_1^\infty\frac{\sin\dfrac{\pi u}{\omega}}{\sin^2\dfrac{n\pi\omega'}{\omega}-\sin^2\dfrac{\pi u}{2\omega}},$$

and therefore, if $\theta = \pi\omega'/2\omega$,

$$\eta' = \frac{\eta\omega'}{\omega} + \frac{\pi}{2\omega}\cot\theta - \frac{\pi}{2\omega}\sum_1^\infty \frac{\sin 2\theta}{\sin^2 2n\theta - \sin^2\theta},$$

and

$$\frac{2}{\pi}(\eta\omega' - \eta'\omega) = \sum_1^\infty \frac{\sin 2\theta}{\sin^2 2n\theta - \sin^2\theta} - \cot\theta$$

$$= \sum_1^\infty \{\cot(2n-1)\theta - \cot(2n+1)\theta\} - \cot\theta$$

$$= -\operatorname{Lim}_{n=\infty} \cot(2n+1)\theta$$

$$= \pm i,$$

according as the imaginary part of θ is positive or negative; that is, according as the real part of $\frac{\omega'}{2i\omega}$ is positive or negative.

The following proof of the expression of $\sigma_1 u$ as an infinite product seems simpler than Mr. Forsyth's.

We have by definition

$$\sigma_1 u = \exp - \eta u \,.\, \frac{\sigma(u+\omega)}{\sigma\omega},$$

and then remembering the definition of σu, and that $\eta = \exp\Sigma\Omega^{-2}$, we get at once

$$\sigma_1 u = \Pi\left(1 - \frac{u}{\Omega_1}\right)\exp\left(\frac{u}{\Omega} + \tfrac{1}{2}\frac{u^2}{\Omega^2}\right)$$

$$= \exp\left(\tfrac{1}{2}\Sigma\frac{u^2}{\Omega^2} - \tfrac{1}{2}\Sigma\frac{u^2}{\Omega_1^{\,2}}\right)\Pi\left(1 - \frac{u}{\Omega_1}\right)\exp\left(\frac{u}{\Omega_1} + \tfrac{1}{2}\frac{u^2}{\Omega_1^{\,2}}\right)$$

$$= \exp(-\tfrac{1}{2}e_1u^2)\,.\,\Pi\left(1 - \frac{u}{\Omega_1}\right)\exp\left(\frac{u}{\Omega_1} + \tfrac{1}{2}\frac{u^2}{\Omega_1^{\,2}}\right).$$

Of course we have to be very careful in working with infinite products, but I think the above is satisfactory.

The Grammar School, Manchester.
May 31st, 1886.

NOTE ON THE SINE-EQUATION IN SPHERICAL TRIGONOMETRY.

By *M. Jenkins, M.A.*

THE fundamental sine-equation $\left(\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} \&\text{c.}\right)$ has a defect from which the equation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

is free, namely that parts between 0 and π may be found which satisfy the former equation, and yet are not necessarily capable of forming a proper spherical triangle.

This defect will be found to be sufficiently supplied by the use of the following relationship.

"In any spherical triangle any side and opposite angle are either of the same affection or are respectively nearer in value to $\frac{1}{2}\pi$ than the remaining sides and the remaining angles of the triangle."

The truth and sufficiency of this relationship may be shown thus:—

Any parts between 0 and π which satisfy the equation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

can form a spherical triangle, because $\cos(b \sim c)$, $\cos a$ and $\cos(b + c)$ would be in descending order of magnitude, and therefore $a > b \sim c$, and $a < b + c$, if $b + c < \pi$, or $a < 2\pi - b - c$, if $b + c > \pi$; and these are the only conditions for being able to form a spherical triangle with three given sides between 0 and π.

The equation in $\sin c$ obtained from

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

is

$$(1 - \sin^2 b \sin^2 A) \sin^2 c - 2 \sin b \cos a \cos A \sin c + \cos^2 a - \cos^2 b = 0.$$

The condition for real roots of this equation is

$$\sin a > \sin b \sin A,$$

a condition which is satisfied by the satisfaction of the equation $\sin a = \frac{\sin b \sin A}{\sin B}$.

We may show in an indirect way that any root of the quadratic which is real must be numerically less than unity; for if $\sin c = F$, $\cos c = \pm\sqrt{(1-F^2)}$ and one or other of these values of $\cos c$, in conjunction with $\sin c = F$, must satisfy the equation $\cos a = \cos b \cos c + \sin b \sin c \cos A$ from which they were derived; and this could not be the case if F were numerically greater than unity, and therefore $\sqrt{(1-F^2)}$ imaginary.

The only remaining conditions for obtaining a real triangle with parts A, a, b is that one or both of the value of $\sin c$ should be positive. But if $\sin c_1$, $\sin c_2$ are these two values,

$$\sin c_1 + \sin c_2 = \frac{2 \sin b \cos a \cos A}{1 - \sin^2 b \sin^2 A}; \quad \sin c_1 \sin c_2 = \frac{\cos^2 a - \cos^2 b}{1 - \sin^2 b \sin^2 A},$$

and therefore either $\cos a \cos A$ is positive, or $\cos^2 a - \cos^2 b$ is negative, that is $\sin^2 a - \sin^2 b$ positive; and from this follows the above stated relationship.

The same result may be shown thus:

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

therefore $$\cos a \cos A = \frac{\cos^2 a - \cos a \cos b \cos c}{\sin b \sin c}.$$

From this and the two similar equations we see that two of the three products $\cos a \cos A$, $\cos b \cos B$, $\cos c \cos C$ must be positive whatever be the order of magnitude of $\cos a$, $\cos b$, $\cos c$: also, if one of the three products, say $\cos a \cos A$, is negative, then since

$$\sin b \sin c . \cos a \cos A = (\cos^2 a - \cos^2 b) + \cos^2 b - \cos a \cos b \cos c,$$

and $\cos^2 b - \cos a \cos b \cos c$ is positive, $\cos^2 a - \cos^2 b$ must be negative.

The figure of the ambiguous case gives the same result.

If $CA < \frac{1}{2}\pi$, $CAA_1 < \frac{1}{2}\pi$, A_1 being the opposite point on the sphere to A, CM the perpendicular from C on AA_1, $C\alpha = CA$, $CE = \frac{1}{2}\pi$; E being the middle point of $A_1\alpha$. As CB passes from CA to CM decreasing, and then through $C\alpha$ to CE, it is of the same affection as CA. It then passes from CE to CA_1, and is of opposite affection to CA, but is nearer in position and magnitude to CE than CA_1 is, and therefore nearer in magnitude to $\frac{1}{2}\pi$ than CA is. The relationship of common affection and nearness to $\frac{1}{2}\pi$ is unaffected by the simultaneous substitution of supplementary values for side and opposite angle; and therefore by means of the three associated or colunar triangles of ABC, the required relationship holds good in the other three cases when $b>\frac{1}{2}\pi$, $A<\frac{1}{2}\pi$; when $b<\frac{1}{2}\pi$, $A>\frac{1}{2}\pi$; and when $b>\frac{1}{2}\pi$, $A>\frac{1}{2}\pi$.

One other mode of writing and proving the relationship may be noticed.

If $S(x)$ denote the sign of x, since $a-b$ is of the same sign as $A-B$ and $a+b-\pi$ of the same sign as $A+B-\pi$, therefore

$$S\{(\tfrac{1}{2}\pi-b)+(\tfrac{1}{2}\pi-a)\}=S\{(\tfrac{1}{2}\pi-B)+(\tfrac{1}{2}\pi-A)\},$$

and $S\{(\frac{1}{2}\pi-b)-(\frac{1}{2}\pi-a)\}=S\{(\frac{1}{2}\pi-B)-(\frac{1}{2}\pi-A)\}$;

therefore

$$S\{(\tfrac{1}{2}\pi-b)^2-(\tfrac{1}{2}\pi-a)^2\}=S\{(\tfrac{1}{2}\pi-B)^2-(\tfrac{1}{2}\pi-A)^2\};$$

whence $\frac{1}{2}\pi\sim b><\frac{1}{2}\pi\sim a$ as $\frac{1}{2}\pi\sim B><\frac{1}{2}\pi\sim A$.

If we suppose $\frac{1}{2}\pi\sim b>\frac{1}{2}\pi\sim a$ and $\frac{1}{2}\pi\sim B>\frac{1}{2}\pi\sim A$, then

$$S\{(\tfrac{1}{2}\pi-b)+(\tfrac{1}{2}\pi-a)\}=S(\tfrac{1}{2}\pi-b),$$

and $$S\{(\tfrac{1}{2}\pi-B)+(\tfrac{1}{2}\pi-A)\}=S(\tfrac{1}{2}\pi-B);$$

therefore $$S(\tfrac{1}{2}\pi-b)=S(\tfrac{1}{2}\pi-B);$$

but $S(\frac{1}{2}\pi-a)$ may $=\pm S(\frac{1}{2}\pi-A)$. Then, it will be found, there is at least one triangle having the parts A, a, b if

$$(\tfrac{1}{2}\pi-a)\,S(\tfrac{1}{2}\pi-A)+(\tfrac{1}{2}\pi\sim b)$$

is positive, and two triangles having the parts A, a, b if

$$(\tfrac{1}{2}\pi-a)\,S(\tfrac{1}{2}\pi-A)-(\tfrac{1}{2}\pi\sim b)$$

is positive.

ON A GEOMETRICAL INTERPRETATION OF THE ALGEBRAICAL EXPRESSION WHICH EQUATED TO ZERO, REPRESENTS A CURVE OR A SURFACE.

By *H. M. Taylor, M.A.*

In text books on Analytical Geometry we find the following theorems enunciated and proved:—

I. If $L=0$ be the equation of a $\left\{\begin{matrix}\text{straight line}\\ \text{plane}\end{matrix}\right\}$, then L represents the perpendicular from any point on the $\left\{\begin{matrix}\text{straight line}\\ \text{plane}\end{matrix}\right\}$ $L=0$, multiplied by a constant.

II. If $S=0$ be the equation of a $\left\{\begin{matrix}\text{circle}\\ \text{sphere}\end{matrix}\right\}$, then S represents the square of the tangent (real or imaginary) drawn from any point to the $\left\{\begin{matrix}\text{circle}\\ \text{sphere}\end{matrix}\right\}$ $S=0$, multiplied by some constant.

Between these two theorems stated as above, there is no apparent analogy; but let the second theorem be stated as follows:—

If $S=0$ be the equation of a $\left\{\begin{matrix}\text{circle}\\ \text{sphere}\end{matrix}\right\}$, then S represents the product of the normals drawn from any point to the $\left\{\begin{matrix}\text{circle}\\ \text{sphere}\end{matrix}\right\}$ $S=0$, multiplied by some constant.

There is now a strong analogy between the two theorems, which naturally suggests the possibility of a mere general theorem which shall embrace both results.

An attempt to generalise these theorems has led to an extension, at all events to the case of central conics and quadrics.

The corresponding theorem for them may be stated thus:—

If $S=0$ be the equation of a $\left\{\begin{matrix}\text{conic}\\ \text{quadric}\end{matrix}\right\}$, then S represents the product of the normals drawn from any point to the $\left\{\begin{matrix}\text{conic}\\ \text{quadric}\end{matrix}\right\}$ $S=0$ and the perpendiculars from the centre of the $\left\{\begin{matrix}\text{conic}\\ \text{quadric}\end{matrix}\right\}$ upon the tangent $\left\{\begin{matrix}\text{lines}\\ \text{planes}\end{matrix}\right\}$ to the $\left\{\begin{matrix}\text{conic}\\ \text{quadric}\end{matrix}\right\}$ at the points to which these normals are drawn, multiplied by some constant.

This proposition is capable of still further generalisation, although to what extent I have not been able to decide. I have succeeded in showing that it is true for some curves of higher degree.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (1) be the equation of an ellipse: then the equation of the normal at any point (x, y) of the curve may be written in the form

$$\frac{\xi - x}{\dfrac{x}{a^2}} = \frac{\eta - y}{\dfrac{y}{b^2}}.$$

If n be the length of the normal at (x, y) which is drawn from the point (ξ, η), and p be the length of the perpendicular from the origin on the tangent to the ellipse at the point (x, y), then

$$n^2 = (\xi - x)^2 + (\eta - y)^2,$$

and
$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4}.$$

Therefore
$$\frac{\xi - x}{\dfrac{x}{a^2}} = \frac{\eta - y}{\dfrac{y}{b^2}} = np.$$

Then
$$\xi = x\left(1 + \frac{np}{a^2}\right),$$

and
$$\eta = y\left(1 + \frac{np}{b^2}\right),$$

whence, substituting in equation (1), we obtain

$$\frac{\xi^2}{a^2\left(1 + \dfrac{np}{a^2}\right)^2} + \frac{\eta^2}{b\left(1 + \dfrac{np}{b^2}\right)^2} = 1;$$

whence, if n_1p_1, n_2p_2, n_3p_3, n_4p_4 be the four roots of this equation,

$$n_1p_1n_2p_2n_3p_3n_4p_4 = a^4b^4\left(1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}\right).$$

In the same manner, it can be shown that, if n be the length of a normal at the point (x, y, z) to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

drawn from the point (ξ, η, ζ), and p be the perpendicular from the centre of the ellipsoid on the tangent plane at (x, y, z), the equation which gives the six values of np is

$$\frac{\xi^2}{a^2\left(1 + \dfrac{np}{a^2}\right)^2} + \frac{\eta^2}{b^2\left(1 + \dfrac{np}{b^2}\right)^2} + \frac{\zeta^2}{c^2\left(1 + \dfrac{np}{c^2}\right)^2} = 1,$$

and that therefore the product of the six roots of this equation in np

$$n_1p_1n_2p_2n_3p_3n_4p_4n_5p_5n_6p_6 = a^4b^4c^4\left(1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2}\right).$$

EXTENSIONS OF FOURIER'S TRIGONOMETRIC SERIES THEOREMS.

By *Peter Alexander.*

THE following theorems are extensions of Fourier's theorems for the expansion of functions in Trigonometric Series.

Putting

$$\sigma(n^2) = \left[1 - \left(\frac{n}{n_1}\right)^2\right]\left[1 - \left(\frac{n}{n_2}\right)^2\right] \cdots,$$

the theorems are as follows.

I. If $\phi(x)$ be such that

$$\phi(-x) = -\phi(x),$$

$$\phi(x) = C_1 \sin n_1 x + C_2 \sin n_2 x + \&c.,$$

where

$$C_\iota = -\frac{2\pi}{n_\iota^3 X^2 \sigma'(n_\iota^2)} \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1 - \left(\frac{\kappa\pi}{n_\iota X}\right)^2} \int_0^X \phi(q) \sin\frac{\kappa\pi q}{X}\, dq,$$

and n_1, n_2, &c. are infinite in number and may be either quite arbitrary or may be derived from some such equation as

$$\frac{nX}{\tan nX} = 1 - hX, \quad \text{(Fourier, § 292).}$$

II. If
$$\psi(-x) = \psi(x),$$

$$\psi(x) = C_0 + C_1 \cos n_1 x + C_2 \cos n_2 x + \&c,$$

where

$$C_0 = \frac{1}{X}\int_0^X \psi(q)\, dq + \frac{2}{X} \sum_{\kappa=1}^{\kappa=\infty} \sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right] \int_0^X \psi(q) \cos\frac{\kappa\pi q}{X}\, dq,$$

and

$$C_l = -\frac{2\pi^2}{n_l^4 X^3 \sigma'(n_l^2)} \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa^2 \sigma\left[\left(\frac{\kappa\pi}{X}\right)^3\right]}{1-\left(\frac{\kappa\pi}{n_l X}\right)^2} \int_0^X \psi(q) \cos\frac{\kappa\pi q}{X}\, dq,$$

where n_1, n_2, &c. and σ have the same meaning as in Theorem I.

III. If $f(x)$ be a function of x, which does not satisfy either

$$f(-x) = -f(x),$$

$$\text{or } f(-x) = f(x),$$

then

$$f(\sigma) = A_0 + A_1 \cos n_1 x + A_2 \cos n_2 x + \&\text{c.}$$
$$+ B_1 \sin n_1 x + B_2 \sin n_2 x + \&\text{c.},$$

where

$$A_0 = \frac{1}{2X}\int_{-X}^{X} f(q)\,dq + \frac{1}{X}\sum_{\kappa=1}^{\kappa=\infty} \sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]\int_{-X}^{X} f(q) \cos\frac{\kappa\pi q}{X}\, dq,$$

$$A_l = -\frac{\pi^2}{X^3 n_l^4 \sigma'(n_l^2)} \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa^2 \sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_l X}\right)^2} \int_{-X}^{X} f(q) \cos\frac{\kappa\pi q}{X}\, dq,$$

and

$$B_l = -\frac{\pi}{X^2 n_l^3 \sigma'(n_l^2)} \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa \sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_l X}\right)^2} \int_{-X}^{X} f(q) \sin\frac{\kappa\pi q}{X}\, dq.$$

Proof of Theorem I.

$$\phi(x) = C_1 \sin n_1 x + C_2 \sin n_2 x + \&\text{c.},$$

therefore

$$\begin{aligned}
\phi\ (0) &= 0,\\
\phi'\ (0) &= n_1 C_1 + n_2 C_2 + \&\text{c.},\\
\phi''\ (0) &= 0,\\
-\phi'''\ (0) &= n_1^3 C_1 + n_2^3 C_2 + \&\text{c.},\\
\phi^4\ (0) &= 0,\\
\phi^5\ (0) &= n_1^5 C_1 + n_2^5 C_2 + \&\text{c.},\\
\phi^6\ (0) &= 0,\\
-\phi^7\ (0) &= n_1^7 C_1 + n_2^7 C_2 + \&\text{c.},\\
\&\text{c.} \qquad & \qquad \&\text{c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).
\end{aligned}$$

Now the theory of determinants shews that if

$$\left.\begin{array}{l}
n_1 y_1 + n_2 y_2 + n_3 y_3 + \ldots + n_m y_m = D\\
n_1^3 y_1 + n_2^3 y_2 + n_3^3 y_3 + \ldots + n_m^3 y_m = D^3\\
n_1^5 y_1 + n_2^5 y_2 + n_3^5 y_3 + \ldots + n_m^5 y_m = D^5\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
n_1^{2m-1} y_1 + n_2^{2m-1} y_2 + n_3^{2m-1} y_3 + \ldots + n_m^{2m-1} y_m = D^{2m-1}
\end{array}\right\}\ldots(b),$$

$$y_l = \frac{\Delta_l}{\Delta},$$

where

$$\Delta_l = \begin{vmatrix}
n_1, & n_2, & \ldots, & n_{l-1}, & D, & n_{l+1}, & \ldots, & n_m\\
n_1^3, & n_2^3, & \ldots, & n_{l-1}^3, & D^3, & n_{l+1}, & \ldots, & n_m^3\\
n_1^5, & n_2^5, & \ldots, & n_{l-1}^5, & D^5, & n_{l+1}^5, & \ldots, & n_m^5\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
n_1^{2m-1}, & n_2^{2m-1}, & \ldots, & n_{l-1}^{2m-1}, & D^{2m-1}, & n_{l+1}^{2m}, & \ldots, & n_m^{2m-1}
\end{vmatrix},$$

and

$$\Delta = \begin{vmatrix}
n_1, & n_2, & \ldots, & n_m\\
n_1^3, & n_2^3, & \ldots, & n_m^3\\
n_1^5, & n_2^5, & \ldots, & n_m^5\\
\vdots & \vdots & & \vdots\\
n_1^{2m-1}, & n_2^{2m-1}, & \ldots, & n_m^{2m-1}
\end{vmatrix}$$

$$= \pm n_2^2 . n_2 \ldots n_m \left[(n_1^2 - n_2^2)(n_1^2 - n_3^2)\ldots\ldots\right.$$

$$\left.\ldots(n_1^2 - n_m^2)\right]\left[(n_2^2 - n_3^2)(n_2^2 - n_4^2)\ldots(n_2^2 - n_m^2)\right]\ldots\left[(n_{m-1}^2 - n_m^2)\right].$$

Now the value of Δ_ι is evidently obtainable from that of Δ by substituting D for n_ι. Hence

$$y_\iota = \frac{D\,(n_1^{\,2}-D^2)\,(n_2^{\,2}-D^2)\ldots(n^2_{\iota-1}-D^2)\,(n^2_{\iota+1}-D^2)\ldots(m_m^{\,2}-D^2)}{n_\iota\,(n_1^{\,2}-n_\iota^{\,2})\,(n_2^{\,2}-n_\iota^{\,2})\ldots(n^2_{\iota-1}-n_\iota^{\,2})\,(n^2_{\iota+1}-n_\iota^{\,2})\ldots(n_m^{\,2}-n_\iota^{2})}\,.$$

Now this is true however great m may be. Let then $m=\infty$ and putting

$$\sigma\,(n^2) = \left[1-\left(\frac{n}{n_1}\right)^2\right]\left[1-\left(\frac{n}{n_2}\right)^2\right]\ldots.$$

Then
$$y_\iota = \frac{D\left[\dfrac{\sigma\,(D^2)}{1-\left(\dfrac{D}{n_\iota}\right)^2}\right]}{n_\iota\left[\dfrac{\sigma\,(n^2)}{1-\left(\dfrac{n}{n_\iota}\right)^2}\right]_{n=n_\iota}}\,,$$

But
$$\left[\frac{\sigma\,(n^2)}{1-\left(\dfrac{n}{n_\iota}\right)^2}\right]_{n=n_\iota} = \left[\frac{\dfrac{d}{dn}\,\sigma\,(n^2)}{\dfrac{d}{dn}\left\{1-\left(\dfrac{n}{n_\iota}\right)^2\right\}}\right]_{n=n_\iota}$$

$$= \left[\frac{\sigma'\,(n^2)\,.\,2n}{-\dfrac{2n}{n_\iota^{\,2}}}\right]_{n=n_\iota}$$

$$= -\,n_\iota^{\,2}\sigma'\,(n_\iota^{\,2}).$$

Therefore
$$y_\iota = -\frac{1}{n_\iota^{\,3}\sigma'\,(n_\iota^{\,2})}\,\frac{D\,.\,\sigma\,(D^2)}{1-\left(\dfrac{D}{n_\iota}\right)^2}\,\ldots\ldots\ldots\ldots\ldots(A)$$

Now if $m=\infty$ equations (b) become the same as equations (a) on the substitution of C_ι for y_ι and $(-1)^{S+1}\phi^{2S+1}(0)$ for D^{2S+1}.

Hence from (A) we derive

$$C_\iota = -\frac{1}{n_\iota^{3}\sigma'\,(n_\iota^{2})}\left\{\frac{\dfrac{d}{dp}\,\sigma\left[-\left(\dfrac{d}{dp}\right)^2\right]}{1+\left(\dfrac{1}{n_\iota}\,\dfrac{d}{dp}\right)^2}\,\phi\,(p)\right\}_{p=0}$$

$$= -\frac{1}{n_\iota^{\,3}\sigma'\,(n_\iota^{\,2})}.\left\{\frac{\sigma\left[-\left(\dfrac{d}{dp}\right)^2\right]}{1+\left(\dfrac{1}{n_\iota}\,\dfrac{d}{dp}\right)^2}\,\phi'\,(p)\right\}_{p=0}\,\ldots\ldots\ldots(B).$$

But, by Fourier's theorem,

$$\phi(p) = \frac{2}{X}\sum_{\kappa=1}^{\kappa=\infty} \sin\frac{\kappa\pi p}{X}\int_0^X \phi(q)\sin\frac{\kappa\pi q}{X}\,dq;$$

therefore

$$\phi'(p) = \frac{2\pi}{X^2}\sum_{\kappa=1}^{\kappa=\infty}\kappa\cos\frac{\kappa\pi p}{X}\int_0^X \phi(q)\sin\frac{\kappa\pi q}{X}\,dq.$$

Now (Boole's *Differential Equations*, Chap. XVI., Art. 6., p. 296)

$$f\left[\left(\frac{d}{dx}\right)^2\right]\cos mx = f(-m^2)\cos mx.$$

Hence

$$\left\{\frac{\sigma\left[-\left(\frac{d}{dp}\right)^2\right]}{1+\left(\frac{1}{n_\iota}\frac{d}{dp}\right)^2}\cos\frac{\kappa\pi p}{X}\right\}_{p=0} = \left\{\frac{\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}\cos\frac{\kappa\pi p}{X}\right\}_{p=0}$$

$$= \frac{\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}.$$

Therefore (B) becomes

$$C_\iota = -\frac{2\pi}{X^2 n_\iota^3\sigma'(n_\iota^2)}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa.\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}\int_0^X \phi(q)\sin\frac{\kappa\pi q}{X}\,dq$$

............(I).

Corollary. If $X=\pi$,

$$C_\iota = -\frac{2}{\pi n_\iota^3\sigma'(n_\iota^2)}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa\sigma(\kappa^2)}{1-\left(\frac{\kappa}{n_\iota}\right)^2}\int_0^\pi \phi(q)\sin\kappa q\,dq\text{......(I)*}.$$

Sub-corollary. If $n_\iota = i$ an integer,

$$\frac{\sigma(\kappa^2)}{1-\left(\frac{\kappa}{n_\iota}\right)^2} = 0$$

for all values of κ except i and

$$\frac{\sigma(\kappa^2)}{1-\left(\frac{\kappa}{n_\iota}\right)^2} = -i^2\sigma'(i^2)$$

when $\kappa = i$;

therefore $$C_i = \frac{2}{\pi} \int_0^\pi \phi(q) \sin \kappa q \, dq,$$

which is Fourier's Theorem.

Proof of Theorem II.

Let $$\psi(x) = C_0 + C_1 \cos n_1 x + C_2 \cos n_2 x + \&c.$$

$$= C_0 \cos n_0 x + C_1 \cos n_1 x + C_2 \cos n_2 x + \&c.,$$

where $$n_0 = 0.$$

therefore $$\psi\ (0) = C_0 + C_1 + C_2 + \&c.,$$

$$\psi'\ (0) = 0,$$

$$-\psi''\ (0) = n_0^2 C_0 + n_1^2 C_1 + n_2^2 C_2 + \&c.,$$

$$\psi'''(0) = 0,$$

$$\psi^4\ (0) = n_0^4 C_0 + n_1^4 C_1 + n_2^4 C_2 + \&c.,$$

$$\psi^4\ (0) = 0,$$

$$-\psi^6\ (0) = n_0^6 C_0 + n_1^6 C_1 + n_2^6 C_2 + \&c.,$$

$$\&c. \qquad \qquad \&c. \ldots\ldots\ldots\ldots\ldots\ldots (a)'.$$

From these, by the method of determinants, we obtain

$$C_i = \frac{\Delta_i}{\Delta},$$

where $$\Delta = \begin{vmatrix} 1, & 1, & 1, & \ldots\ldots \\ n_0^2, & n_1^2, & n_2^2, & \ldots\ldots \\ n_0^4, & n_1^4, & n_2^4, & \ldots\ldots \\ n_0^6, & n_1^6, & n_2^6, & \ldots\ldots \\ \vdots & \vdots & \vdots & \vdots\ \vdots\ \vdots \end{vmatrix}$$

and Δ_i is what Δ becomes when $(-)^s \psi^{2s}(0)$ is substituted for n_i^{2s}.

Hence
$$C_i = \left[\frac{\left\{\left[n_0^2+\left(\frac{d}{dp}\right)^2\right]\left[n_1^2+\left(\frac{d}{dp}\right)^2\right]\cdots\left[n^2_{i-1}+\left(\frac{d}{dp}\right)^2\right]\left[n^2_{i+1}+\left(\frac{d}{dp}\right)^2\right]\cdots\right\}\psi(p)}{(n_0^2-n_i^2)(n_1^2-n_i^2)(n_2^2-n_i^2)\ldots(n^2_{i-1}-n_i^2)(n^2_{i+1}-n_i^2)\ldots}\right]_{p=0}$$

$$= -\left[\frac{\left\{\left(\frac{d}{dp}\right)^2\left[n_1^2+\left(\frac{d}{dp}\right)^2\right]\left[n_2^2+\left(\frac{d}{dp}\right)^2\right]\cdots\left[n^2_{i-1}+\left(\frac{d}{dp}\right)^2\right]\left[n^2_{i+1}+\left(\frac{d}{dp}\right)^2\right]\cdots\right\}\psi(p)}{n_i^2(n_1^2-n_i^2)(n_2^2-n_i^2)\ldots[n^2_{i-1}-n^2_i](n^2_{i+1}-n_i^2)\ldots}\right]_{p=0}$$

and
$$C_0 = \left[\frac{\left\{\left[n_1^2+\left(\frac{d}{dp}\right)^2\right]\left[n_2^2+\left(\frac{d}{dp}\right)^2\right]\left[n_3^2+\left(\frac{d}{dp}\right)^2\right]\cdots\right\}\psi(p)}{n_1^2 . n_2^2 . n_3^2 \ldots\ldots}\right]_{p=0}.$$

Putting as before

$$\sigma(n^2) = \left[1-\left(\frac{n}{n_1}\right)^2\right]\left[1-\left(\frac{n}{n_2}\right)^2\right]\ldots\ldots\ldots\ldots$$

$$C_i = -\frac{\left\{\frac{\sigma\left[-\left(\frac{d}{dp}\right)^2\right]}{1+\left(\frac{1}{n_i}\frac{d}{dp}\right)^2}\psi''(p)\right\}_{p=0}}{n_i^2\left\{\frac{\sigma(n^2)}{1-\left(\frac{n}{n_i}\right)^2}\right\}_{n=n_i}} = \frac{1}{n_i^4\sigma''(n_i^2)}\left\{\frac{\sigma\left[-\left(\frac{d}{dp}\right)^2\right]}{1+\left(\frac{1}{n^i}\frac{d}{dp}\right)^2}\psi''(p)\right\}_{p=0}$$

and $$C_0 = \left\{\sigma\left[-\left(\frac{d}{dp}\right)^2\right]\psi(p)\right\}_{p=0}.$$

Now, by Fourier's Theorem,

$$\psi(p) = \frac{1}{X}\int_0^X \psi(q)\,dq + \frac{2}{X}\sum_{\kappa=1}^{\kappa=\infty}\cos\frac{\kappa\pi p}{X}\int_0^X \psi(q)\cos\frac{\kappa\pi q}{X}\,dq;$$

therefore

$$\psi''(p) = -\frac{2\pi^2}{X^3}\sum_{\kappa=1}^{\kappa=\infty}\kappa^2\cos\frac{\kappa\pi p}{X}\int_0^X \psi(q)\cos\frac{\kappa\pi q}{X}\,dq.$$

Now

$$\left[\left\{\frac{\sigma\left[-\left(\frac{d}{dp}\right)^2\right]}{1+\left(\frac{1}{n_\iota}\frac{d}{dp}\right)^2}\right\}\cos\left(\frac{\kappa\pi p}{X}\right)\right]_{p=0} = \left\{\frac{\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}\cos\frac{\kappa\pi p}{X}\right\}_{p=0}$$

$$= \frac{\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2},$$

$$\left\{\sigma\left[-\left(\frac{d}{dp}\right)^2\right]\cos\left(\frac{\kappa\pi p}{X}\right)\right\}_{p=0} = \left\{\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]\cos\left(\frac{\kappa\pi p}{X}\right)\right]_{p=0}$$

$$= \sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right].$$

Therefore

$$C_\iota = -\frac{2\pi^2}{X^3 n_\iota^4\sigma'(n_\iota^2)}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa^2\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}\int_0^X \psi(q)\cos\frac{\kappa\pi q}{X}\,dq$$

and ………(II),

$$C_0 = \frac{1}{X}\int_0^X \psi(q)\,dq + \frac{2}{X}\sum_{\kappa=1}^{\kappa=\infty}\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]\int_0^X \psi(q)\cos\frac{\kappa\pi q}{X}\,dq$$

………(II′).

Corollary. If $X = \pi$,

$$C_\iota = -\frac{2}{\pi n_\iota^4\sigma'(n_\iota^2)}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa^2\sigma[\kappa^2]}{1-\left(\frac{\kappa}{n_\iota}\right)^2}\int_0^X \psi(q)\cos\kappa q\,dq \text{...(II)*},$$

$$C_0 = \frac{1}{\pi}\int_0^X \psi(q)\,dq + \frac{2}{\pi}\sum_{\kappa=1}^{\kappa=\infty}\sigma(\kappa^2)\int_0^X \psi(q)\cos\kappa q\,dq$$

………(II)*.

Sub-corollary. If $n_\iota = \iota$

$$\sigma(\kappa^2) = 0 \quad \text{and} \quad \frac{\sigma(\kappa^2)}{1 - \left(\frac{\kappa}{n_\iota}\right)^2} = 0,$$

except when $\kappa = \iota$, when it $= -\iota^2 \sigma'(\iota^2)$;

$$\left.\begin{aligned} &\text{therefore} \qquad C_0 = \frac{1}{\pi}\int_0^\pi \psi(q)\, dq \\ &\text{and} \qquad C_\iota = \frac{2}{\pi}\int_0^\pi \psi(q) \cos \iota q\, dq \end{aligned}\right\} \textit{Fourier's Theorem.}$$

Proof of Theorem III.

Let $f(\chi)$ be a function of χ which does not satisfy either $f(-\chi) = f(\chi)$ or $f(-\chi) = -f(\chi)$.

Putting $$\psi(\chi) = \frac{f(\chi) + f(-\chi)}{2},$$

and $$\phi(\chi) = \frac{f(\chi) - f(-\chi)}{2};$$

and therefore $$\psi(\chi) + \phi(\chi) = f(\chi).$$

It is evident that

$$\psi(-\chi) = \psi(\chi),$$

and $$\phi(-\chi) = -\phi(\chi).$$

Hence

$$\int_{-X}^{X} \psi(q) \cos\frac{\kappa\pi q}{X}\, dq = 2\int_0^X \psi(q) \cos\frac{\kappa\pi q}{X}\, dq,$$

$$\int_{-X}^{X} \phi(q) \sin\frac{\kappa\pi q}{X}\, dq = 2\int_0^X \phi(q) \sin\frac{\kappa\pi q}{X}\, dq,$$

$$\int_{-X}^{X} \psi(q) \sin\frac{\kappa\pi q}{X}\, dq = 0,$$

and

$$\int_{-X}^{X} \phi(q) \cos\frac{\kappa\pi q}{X}\, dq = 0.$$

Therefore

$$\begin{aligned} \int_{-X}^{X} f(q) \cos\frac{\kappa\pi q}{X}\, dq &= \int_{-X}^{X} \psi(q) \cos\frac{\kappa\pi q}{X}\, dq + \int_{-X}^{X} \phi(q) \cos\frac{\kappa\pi q}{X}\, dq \\ &= 2\int_0^X \psi(q) \cos\frac{\kappa\pi q}{X}\, dq, \end{aligned}$$

and

$$\int_{-X}^{X} f(q)\sin\frac{\kappa\pi q}{X}\,dq = \int_{-X}^{X}\phi(q)\sin\frac{\kappa\pi q}{X}\,dq + \int_{-X}^{X}\psi(q)\sin\frac{\kappa\pi q}{X}\,dq$$

$$= 2\int_{0}^{X}\phi(q)\sin\frac{\kappa\pi q}{X}\,dq.$$

Hence, if

$$\psi(x) = A_0 + A_1\cos n_1 x + A_2\cos n_2 x + \&c.,$$

and
$$\phi(x) = \qquad B_1\sin n_1 x + B_2\sin n_2 x + \&c.,$$

$$A_0 = \frac{1}{X}\int_0^X \psi(q)\,dq + \frac{2}{X}\sum_{\kappa=1}^{\kappa=\infty}\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]\int_0^X \psi(q)\cos\frac{\kappa\pi q}{X}\,dq$$

$$= \frac{1}{2X}\int_{-X}^{X} f(q)dq + \frac{1}{X}\sum_{\kappa=1}^{\kappa=\infty}\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]\int_{-X}^{X} f(q)\cos\frac{\kappa\pi q}{X}\,dq$$

....................(III),

$$A_\iota = -\frac{2\pi^2}{X^3 n_\iota^4\sigma'(n_\iota^2)}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa^2\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}\int_0^X \psi(q)\cos\frac{\kappa\pi q}{X}\,dq$$

$$= -\frac{\pi^2}{X^3 n_\iota^4\sigma'(n_\iota^2)}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa^3\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}\int_{-X}^{X} f(q)\cos\frac{\kappa\pi q}{X}\,dq$$

....................(III).

And

$$B_\iota = -\frac{2\pi}{X^2 n_\iota^3\sigma'(n_\iota^2)}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}\int_0^X \phi(q)\sin\frac{\kappa\pi q}{X}\,dq$$

$$= -\frac{\pi}{X^2 n_\iota^3\sigma'(n_\iota^2)}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1-\left(\frac{\kappa\pi}{n_\iota X}\right)^2}\int_{-X}^{X} f(q)\sin\frac{\kappa\pi q}{X}\,dq$$

...............(III).

These are the values of the coefficients in

$$\begin{aligned} f(\chi) &= \psi(\chi) + \phi(\chi) \\ &= A_0 + A_1 \cos n_1\chi + A_2 \cos n_2\chi + \&c. \\ &\quad + B_1 \sin n_1\chi + B_2 \sin n_2\chi + \&c. \end{aligned}$$

Corollary. If $X = \pi$,

$$\left.\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(q)\,dq + \frac{1}{\pi}\sum_{\kappa=1}^{\kappa=\infty} \sigma(\kappa^2)\int_{-\pi}^{\pi} f(q)\cos\kappa q\,dq \\ A_\iota &= -\frac{1}{\pi n_\iota^4 \sigma'(n_\iota^2)} \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa^2\sigma(\kappa^2)}{1-\left(\frac{\kappa}{n_\iota}\right)^2} \int_{-\pi}^{\pi} f(q)\cos\kappa q\,dq \\ B_\iota &= -\frac{1}{\pi n_\iota^3 \sigma'(n_\iota^2)} \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa^2\sigma(\kappa^2)}{1-\left(\frac{\kappa}{n_\iota}\right)^2} \int_{-\pi}^{\pi} f(q)\sin\kappa q\,dq \end{aligned}\right\}$$

...............(III)*.

Sub-Corollary.

If $$n_\iota = \iota, \quad \sigma(\kappa^2) = 0,$$

and $$\frac{\sigma(\kappa^2)}{1-\left(\frac{\kappa}{n_\iota}\right)^2} = 0,$$

except when $\kappa = \iota$, in which case

$$\frac{\sigma(\kappa^2)}{1-\left(\frac{\kappa}{n}\right)^2} = -\iota^2\sigma'(\iota^2);$$

therefore $$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(q)\,dq,$$

$$A_\iota = \frac{1}{\pi}\int_{-\pi}^{\pi} f(q)\cos iq\,dq,$$

$$B_\iota = \frac{1}{\pi}\int_{-\pi}^{\pi} f(q)\sin iq\,dq,$$

which is Fourier's Theorem.

Corollary 2. If $$n^{\iota} = \frac{\iota\pi}{X},$$

$$\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right] = 0,$$

and $$\frac{\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1 - \left(\frac{\kappa\pi}{n_\iota X}\right)^2} = 0,$$

except when $\kappa = \iota$, in which case

$$\frac{\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]}{1 - \left(\frac{\kappa\pi}{n_\iota X}\right)^2} = -n_\iota^2 \sigma'(n_\iota^2).$$

Hence (III) becomes

$$A_0 = \frac{1}{2X}\int_{-X}^{X} f(q)\,dq,$$

$$A_\iota = \frac{1}{X}\int_{X}^{X} f(q)\cos\frac{i\pi q}{X}\,dq,$$

$$B_\iota = \frac{1}{X}\int_{-X}^{X} f(q)\sin\frac{i\pi q}{X}\,dq,$$

which is Fourier's Theorem.

(I) and (II) similarly treated give the corresponding forms of Fourier's Theorem.

To illustrate what use might be made of (I)* let it be required to expand x in a series of sines of odd multiples of x.

That is, let

$$x = A_1 \sin x + A_2 \sin 3x + A_3 \sin 5x + \&c.$$

Then $$n_1 = 1,\ n_2 = 3,\ n_3 = 5,\ \ldots,\ n_\iota = 2\iota - 1;$$

therefore $$\sigma(n^2) = \left[1 - \left(\frac{n}{n_1}\right)^2\right]\left[1 - \left(\frac{n}{n_2}\right)^2\right]\ldots$$

$$= \left[1 - \left(\frac{n}{1}\right)^2\right]\left[1 - \left(\frac{n}{3}\right)^2\right]\left[1 - \left(\frac{n}{5}\right)^2\right]\ldots$$

$$= \cos\frac{n\pi}{2};$$

therefore $$2n\sigma'(n^2) = -\frac{\pi}{2}\sin\frac{n\pi}{2};$$

therefore $$\sigma'(n^2) = -\frac{\pi}{4n}\sin\frac{n\pi}{2};$$

therefore $$\sigma'(n_\iota^2) = -\frac{\pi}{4(2\iota-1)}\sin\frac{(2\iota-1)\pi}{2}.$$

Hence, from (I)*,

$$C_\iota = \frac{2}{\pi(2\iota-1)^3\,\dfrac{\pi}{4(2\iota-1)}\sin\dfrac{(2\iota-1)\pi}{2}}$$

$$\times\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa\cos\dfrac{\kappa\pi}{2}}{1-\left(\dfrac{\kappa}{2\iota-1}\right)^2}\int_0^\pi q\sin\kappa q\,dq$$

$$= \frac{8}{[(2\iota-1)\pi]^2\sin\dfrac{(2\iota-1)\pi}{2}}\left\{\frac{-2}{1-\left(\dfrac{2}{2\iota-1}\right)^2}\left(-\frac{\pi\cos 2\pi}{2}\right)\right.$$

$$\left.+\frac{4}{1-\left(\dfrac{4}{2\iota-1}\right)^2}\left(\frac{-\pi\cos 4\pi}{4}\right) - \&c.\right\} + \frac{2}{2\iota-1}$$

$$= \frac{8}{\pi\sin\dfrac{(2\iota-1)\pi}{2}}\left\{\frac{1}{(2\iota-1)^2-2^2} - \frac{1}{(2\iota-1)^2-4^2} + \&c.\right\} + \frac{2}{2\iota-1}.$$

Corollary. In Todhunter's *Int. Cal.*, §312, x is expanded similarly by a different method between the limits $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This result is

$$C_\iota = \frac{4}{(2\iota-1)^2\pi\sin\dfrac{(2\iota-1)\pi}{2}}.$$

Equating these values of C_ι we have

$$\frac{1}{(2\iota-1)^2-2^2} - \frac{1}{(2\iota-1)^2-4^2} + \frac{1}{(2\iota-1)^2-6^2} - \&c.$$

$$= \frac{1}{2(2\iota-1)^2} - \frac{\pi\sin\dfrac{(2\iota-1)\pi}{2}}{4(2\iota-1)},$$

or

$$\frac{1}{(2\iota+1)(2\iota-3)}-\frac{1}{(2\iota+3)(2\iota-5)}+\frac{1}{(2\iota+5)(2\iota-7)}-\text{\&c.}$$

$$=\frac{1}{2(2\iota-1)^2}-\frac{\pi\sin\dfrac{(2\iota-1)\pi}{2}}{4(2\iota-1)},$$

which gives the means of summing all infinite series of the form

$$\frac{1}{[2(m-\iota)+1][2(m+\iota)-1]}-\frac{1}{[2(m-\iota)+3][2(m+\iota)+1]}$$

$$+\frac{1}{[2(m-\iota)+5][2(m+\iota)+3]}-\text{\&c.}$$

Numerous similar results may be obtained by similar comparisons of the values of C_ι obtained by different methods.

An interesting example of the application of (I) is to expand $\phi(x)$ as follows:

$$\phi(\chi)=C_1\sin n_1x+C_2\sin n_2x+\text{\&c.},$$

where n_1, n_2, &c. are the roots of

$$\frac{nX}{\tan nX}=1-hX\ \text{(Fourier, § 292).}$$

It is easily seen that

$$\sigma(n^2)=\frac{nX\cos nX-(1-hX)\sin nX}{nhX^2};$$

therefore

$$\sigma'(n^2)=\frac{(1-hX)(\sin nX-nX\cos nX)-n^2X^2\sin nX}{2n^3hX^2};$$

therefore $$\sigma'(n_\iota^2)=\frac{h\cos n_\iota X-n_\iota\sin n_\iota X}{2hn_\iota^2}.$$

Also $$\sigma\left[\left(\frac{\kappa\pi}{X}\right)^2\right]=\frac{\kappa\pi\cos\kappa\pi-(1-hX)\sin\kappa\pi}{\kappa\pi Xh}$$

$$=\frac{\cos\kappa\pi}{Xh},\ \text{because } \kappa \text{ is an integer.}$$

Hence (I) becomes

$$C_\iota = -\frac{4\pi}{n_\iota X^3 (h \cos n_\iota X - n_\iota \sin n_\iota X)}$$
$$\times \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa \cos \kappa\pi}{1 - \left(\frac{\kappa\pi}{n_\iota X}\right)^2} \int_0^X \phi(q) \sin \frac{\kappa\pi q}{X}\, dq$$

$$= \frac{8\pi \sin n_\iota X}{n_\iota X^2 (2n_\iota X - \sin 2n_\iota X)}$$
$$\times \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa \cos \kappa\pi}{1 - \left(\frac{\kappa\pi}{n_\iota X}\right)^2} \int_0^X \phi(q) \sin \frac{\kappa\pi q}{X}\, dq.$$

Cor. Fourier gives us the solution

$$C_\iota = \frac{2}{X - \frac{1}{2n_\iota} \sin 2n_\iota X} \int_0^X \phi(q) \sin n_\iota q\, dq.$$

Equating these we obtain

$$\int_0^X \phi(q) \sin n_\iota q\, dq = \frac{2\pi \sin n_\iota X}{(n_\iota X)^2}$$
$$\times \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa \cos \kappa\pi}{1 - \left(\frac{\kappa\pi}{n_\iota X}\right)^2} \int_0^X \phi(q) \sin \frac{\kappa\pi q}{X}\, dq;$$

therefore

$$\sin n_\iota q = 2\pi \sin n_\iota X \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa \cos \kappa\pi}{(n_\iota X)^2 - (\kappa\pi)^2} \sin \frac{\kappa\pi q}{X}.$$

If $X = \pi$, this becomes

$$\sin n_\iota q = \frac{2}{\pi} \sin n_\iota \pi \sum_{\kappa=1}^{\kappa=\infty} \frac{\kappa \cos \kappa\pi}{n_\iota^2 - \kappa^2} \sin \kappa q,$$

a known result (Todhunter, *Int. Cal.* § 316).

The forms (B) and (B') or modifications thereof will sometimes be found more convenient than (I.) and (II.). (B) is equivalent to

$$C_\iota = \frac{\phi'(0) + P_\iota \phi'''(0) + Q_\iota \phi^5(0) + \&c.}{n_\iota (1 - P_\iota n_\iota^2 + Q_\iota n_\iota^4 - \&c.)},$$

or

$$\frac{\phi'(0) + P_\iota \phi'''(0) + \&c.}{-n_\iota^3 \sigma'(n_\iota^2)} \quad \ldots\ldots\ldots\ldots (E),$$

where $P_\iota = \frac{1}{n_1^2} + \frac{1}{n_2^2} + \ldots + \frac{1}{n^2_{\iota-1}} + \frac{1}{n^2_{\iota+1}} + \ldots$, $\frac{1}{n_\iota^2}$ being omitted,

$Q_\iota = \frac{1}{n_1^2 n_2^2} + \frac{1}{n_1^2 n_3^2} + \&c.$

= the sum of all the products two at a time of $\frac{1}{n_1^2}$, $\frac{1}{n_2^2}$, &c.,

$\frac{1}{n_\iota^2}$ being omitted, &c.

Taking the first illustration again, viz.

$$\phi(x) = x = C_1 \sin x + C_2 \sin 3x + C_3 \sin 5x + \&c.,$$

therefore $\phi'(0) = 1, \quad \phi'''(0) = 0, \quad \phi^5(0) = 0, \quad \&c.$

and $$\sigma(n^2) = \cos\left(\frac{n\pi}{2}\right),$$

therefore $$\sigma'(n^2) = -\frac{\pi}{4n} \sin \frac{n\pi}{2};$$

therefore $$\sigma'(n_\iota^2) = -\frac{\pi}{4(2\iota - 1)} \sin \frac{(2\iota - 1)\pi}{2};$$

therefore $$C_\iota = \frac{1}{(2\iota - 1)^3 . \frac{\pi}{4(2\iota - 1)} \sin \frac{(2\iota - 1)\pi}{2}}$$

$$= \frac{4}{(2\iota - 1)^2 \pi \sin \frac{(2\iota - 1)\pi}{2}},$$

therefore $$x = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \&c.\right).$$

Although in strictness n_1, n_2, &c., must be infinite in number, yet (E) will give approximate results if n_1, n_2, &c., be taken finite in number, the approximation being closer the more numerous they are taken.

Let, for example, $n_1 = 1$ and $n_2 = 2$, and let it be required to express $\sin mx$ approximately, thus

$$\sin mx = C_1 \sin x + C_2 \sin 2x.$$

Then from (E)

$$C_1 = \frac{m \cos 0 - P_1 . m^3 \cos 0}{1(1 - P_1 . 1^2)} = \frac{m - \frac{1}{2^2} . m^3}{1 - \frac{1}{2^2}} = \frac{4m - m^3}{3},$$

and $$C_2 = \frac{m\cos 0 - P_2 . m^3 \cos 0}{2\,(1 - P_1 . 2^2} = \frac{m - \frac{1}{1^2} . m^3}{2\left(1 - \frac{1}{1^2} . 2^2\right)} = -\frac{m - m^3}{6};$$

therefore

$$\sin mx = \frac{4m - m^3}{3}\sin x - \frac{m - m^3}{6}\sin 2x,$$

which will be found to be very approximately correct for small values of m and x.

The modified form of (B') is

$$\left.\begin{array}{ll} & C_\iota = \dfrac{\psi''(0) + P_\iota\psi^4(0) + Q_\iota\psi^6(0) + \&c.}{n_\iota^{\,4}\sigma'(n_\iota^{\,2})} \\ \text{or} & \quad - \dfrac{\psi''(0) + P_\iota\psi^4(0) + Q_\iota\psi^6(0) + \&c.}{n_\iota^{\,2}(1 - P_\iota n_\iota^{\,2} + Q_\iota n_\iota^{\,4} - \&c.)} \\ \text{and} & C_0 = \psi(0) + P\psi''(0) + Q\psi^4(0) + \&c. \end{array}\right\}\cdots\cdots(E'),$$

where $P = \frac{1}{n_1^{\,2}} + \frac{1}{n_2^{\,2}} + \&c.,\quad Q = \frac{1}{n_1^{\,2}n_2^{\,2}} + \&c.,\quad \&c.,$

with no omissions.

Similarly, if

$$f(x) = A_0 + A_1\cos n_1x + A_2\cos n_2x + \&c.$$
$$+ B_1\sin n_1x + B_2\sin n_2x + \&c.,$$

$$\left.\begin{array}{l} A_0 = \quad f(0) + Pf''(0) + Qf^4(0) + \&c. \\ A_\iota = \quad \dfrac{1}{n_\iota^{\,4}\sigma'(n_\iota^{\,2})}\{f''(0) + P_\iota f^4(0) + Q_\iota f^6(0) + \&c.\} \\ \text{or} \\ \quad = -\dfrac{f''(0) + P_\iota f^4(0) + Q_\iota f^6(0) + \&c.}{n_\iota^{\,2}\{1 - P_\iota n_\iota^{\,2} + Q_\iota n_\iota^{\,4} - \&c.\}} \\ \text{and} \\ B_\iota = -\dfrac{1}{n_\iota^{\,3}\sigma'(n_\iota^{\,2})}\{f'(0) + P_\iota f'''(0) + Q_\iota f^5(0) + \&c.\} \\ \text{or} \\ \quad \dfrac{f'(0) + P_\iota f'''(0) + Q_\iota f^5(0) + \&c.}{n_\iota\{1 - P_\iota n_\iota^{\,2} + Q_\iota n^{\iota 4} - \&c.\}} \end{array}\right\}\cdots(E'').$$

Lord Rayleigh, in his treatise on *Sound*, § 31, says "Fourier's theorem is not obvious. A vague notion is not uncommon that the infinitude of arbitrary constants in the series of necessity endows it with the capacity of representing an arbitrary periodic function. That this is an error will be apparent when it is observed that the same argument would apply equally if one term of the series were omitted, in which case the expansion would not in general be possible."

This argument, though seemingly plausible, may be shown inapplicable by effecting the expansion indicated. This can be done by (I)* as follows:

Let it be required to expand $\phi(x)$ in a series of sines of multiples of x, omitting that with $\sin 3x$.

Here $n_1 = 1$, $n_2 = 2$, $n_3 = 4$, $n_4 = 5$, ...; therefore

$$\sigma(n^2) = \left\{1 - \left(\frac{n}{1}\right)^2\right\} \left\{1 - \left(\frac{n}{2}\right)^2\right\} \left\{1 - \left(\frac{n}{4}\right)^2\right\} \left\{1 - \left(\frac{n}{5}\right)^2\right\} \ldots$$

$$= \frac{\sin n\pi}{n\pi \left\{1 - \left(\frac{n}{3}\right)^2\right\}};$$

therefore

$$2n\sigma'(n^2) = \frac{1}{\pi} \left[\frac{n\left\{1 - \left(\frac{n}{3}\right)^2\right\} \pi \cos n\pi - \left(1 - \frac{n^2}{3}\right) \sin n\pi}{n^2 \left\{1 - \left(\frac{n}{3}\right)^2\right\}^2}\right];$$

$$\sigma'(n_\iota^2) = \frac{\cos n_\iota \pi}{2n_\iota^2 \left\{1 - \left(\frac{n_\iota}{3}\right)^2\right\}}.$$

Also $\sigma(\kappa^2) = \dfrac{\sin \kappa\pi}{\kappa\pi\left\{1 - \left(\frac{\kappa}{3}\right)^2\right\}} = 0$, except when $\kappa = 3$,

$$= \frac{1}{3\pi} \left(\frac{\pi \cos \kappa\pi}{-\frac{2\kappa}{9}}\right)_{\kappa=3}, \text{ when } \kappa = 3,$$

$$= \tfrac{1}{2} \qquad \text{,,} \qquad \text{,,}$$

and
$$\frac{\sigma(\kappa^2)}{1-\left(\frac{\kappa}{n_\iota}\right)^2}=0, \text{ except when } \kappa=3 \text{ or } n_\iota,$$

$$=\frac{1}{2\left\{1-\left(\frac{3}{n_\iota}\right)^2\right\}}, \text{ when } \kappa=3,$$

$$=\left\{\frac{2\kappa\sigma'(\kappa^2)}{-\frac{2\kappa}{n_\iota^2}}\right\}_{\kappa=n_\iota}, \text{ when } \kappa=n_\iota,$$

$$=-n_\iota^2\sigma'(n_\iota^2) \qquad \text{"} \qquad \text{"}$$

Hence (I)* becomes

$$C_\iota=\frac{2}{\pi n_\iota^3\sigma'(n_\iota^2)}\left\{\frac{3}{2\left[1-\left(\frac{3}{n_\iota}\right)^2\right]}\int_0^\pi \phi(q)\sin 3q\,dq\right.$$

$$\left.-n_\iota^3\sigma'(n_\iota^2)\int_0^\pi \phi(q)\sin n_\iota q\,dq\right\}$$

$$=\frac{2}{\pi}\int_0^\pi \phi(q)\sin n_\iota q\,dq-\frac{3}{\pi n_\iota^3\sigma'(n_\iota^2)\left[1-\left(\frac{3}{n_\iota}\right)^2\right]}$$

$$\times\int_0^\pi \phi(q)\sin 3q\,dq$$

$$=\frac{2}{\pi}\int_0^\pi \phi(q)\sin n_\iota q\,dq-\frac{6\left[1-\left(\frac{n_\iota}{3}\right)^2\right]}{n_\iota\pi\left[1-\left(\frac{3}{n_\iota}\right)^2\right]\cos n_\iota\pi}$$

$$\times\int_0^\pi \phi(q)\sin 3q\,dq$$

$$=\frac{2}{\pi}\int_0^\pi \phi(q)\sin n_\iota q\,dq+\frac{2n_\iota}{3\pi\cos n_\iota\pi}\int_0^\pi \phi(q)\sin 3q\,dq\,;$$

therefore

$$\begin{aligned}\phi(x) = \frac{2}{\pi}\Bigg\{&\left[\int_0^\pi \phi(q)\sin q\;dq - \tfrac{1}{3}\int_0^\pi \phi(q)\sin 3q\,dq\right]\sin x\\ &+\left[\int_0^\pi \phi(q)\sin 2q\,dq + \tfrac{2}{3}\int_0^\pi \phi(q)\sin 3q\,dq\right]\sin 2x\\ &+\left[\int_0^\pi \phi(q)\sin 4q\,dq + \tfrac{4}{3}\int_0^\pi \phi(q)\sin 3q\,dq\right]\sin 4x\\ &+\left[\int_0^\pi \phi(q)\sin 5q\,dq - \tfrac{5}{3}\int_0^\pi \phi(q)\sin 3q\,dq\right]\sin 5x\\ &+\text{\&c.}\Bigg\},\end{aligned}$$

which is the required expansion.

That this is a true expansion may be verified thus:—To the sum within the parentheses add the zero term

$$\left\{\int_0^\pi \phi(q)\sin 3q\,dq - \tfrac{3}{3}\int_0^\pi \phi(q)\sin 3q\,dq\right\}\sin 3x,$$

and we have

$$\begin{aligned}\phi(x) &= \frac{2}{\pi}\sum_{\iota=1}^{\iota=\infty}\sin \iota x\int_0^\pi \phi(q)\sin \iota q\,dq\\ &\quad-\frac{2}{3\pi}\{\sin x - 2\sin 2x + 3\sin 3x - 4\sin 4x + \text{\&c.}\}\int_0^\pi \phi(q)\sin 3q\,dq\\ &= \phi(x) - \frac{2}{3\pi}\{\sin x - 2\sin 2x + 3\sin 3x - \text{\&c.}\}\int_0^\pi \phi(q)\sin 3q\,dq\,;\end{aligned}$$

therefore $\sin x - 2\sin 2x + 3\sin 3x - \text{\&c.} = 0,$

a known result.

Fourier's theorems are easily deducible from the forms (E), (E'), and (E'').

ON DOUBLE ALGEBRA.

By *Arthur Buchheim, M.A.*

The problem of multiple Algebra, as solved by Peirce, consisted in the formation of the multiplication tables for the different kinds of Algebra. There is a fairly obvious and perfectly definite connexion between Peirce's linear associative Algebra and the theory of matrices, which it is not worth while to work out here. In the present note I solve, for the case of binary matrices, the problem of finding the n-ary matrices that can be units in an r-tuple Algebra, r being not greater than n. There are two kinds of double Algebra, according as the Algebra does or does not contain the matrix unity: if an Algebra contains the matrix unity we can take any binary matrix as its second unit, and all the conditions of the problem are satisfied. Now, considering the case in which an Algebra does not contain the matrix unity, it can be proved that all the matrices of the Algebra must be vacuous. For let A, B be two matrices of the system and let ω denote the matrix unity; let α be a non-vacuous matrix of the system. Then we have a linear relation connecting α^2, α, ω; therefore we have a linear relation connecting A, B, ω; that is, ω is a matrix of the system.

Now let

$$A = \begin{matrix} a, & b, \\ c, & d, \end{matrix} \qquad B = \begin{matrix} a', & b', \\ c', & d, \end{matrix}$$

then the conditions that every matrix of the form $\lambda A + \mu B$ may be vacuous are

$$0 = ad - bc = ad' + a'd - bc' - b'c = a'd' - b'c'.$$

Now by a proper choice of the points of reference (*v. Proc. L. M. S.* XVI. 73) we can reduce A to the form

$$A = \begin{matrix} 0, & b, \\ 0, & d', \end{matrix}$$

and then we must have

$$a'd - bc' = 0,$$

and we can assume $b' = d' = 0$,[*] and then we must have, to a scalar factor près,

$$B = \begin{matrix} b, & 0, \\ d, & 0, \end{matrix}$$

and now it is easy to verify that

$$A^2 = dA,$$

$$B^2 = dB,$$

$$AB = aB,$$

$$BA = bA,$$

and therefore we conclude that there are two kinds of double Algebra:

(1) That derived from any matrix, and the matrix unity.

(2) That derived from any two matrices such that

$$\mathrm{Det}\,(\lambda A - \mu B)$$

vanishes identically.

The Grammar School, Manchester,
July 15*th*, 1886.

NOTE ON THE QUADRIC AND THE CUBIC.

By *A. R. Johnson.*

The following results were suggested by Art. 235 of Salmon's *Geometry of Three Dimensions*, Ed. 4.

By counting constants, the following table is constructed:

Quantic	3-ary.			4-ary.		5-ary.		6-ary.	
No. of Quantics	2	3	4	2	3	2	3	2	3
Quadric	3*	4	4*	4*	5*	5*	7	7	8
Cubic	5*	6*	7	8*	10*	12	15*	16*	21*

This table shews, for instance, that 3 conics have a common self-conjugate quadrilateral, and 3 quadrics a common self-conjugate pentahedron, and so on; the asterisk denoting that there is only a finite number of such self-conjugate polyhedra.

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[*] We take for e_1 the null point of A, and for e_2 the null point of B.

I. (1). Three *Conics* are stated by the table to have an infinity of self-conjugate quadrilaterals. Calling them

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta u^2 = 0,$$
$$\alpha' x^2 + \beta' y^2 + \gamma' z^2 + \delta u^2 = 0,$$
$$\alpha'' x^2 + \beta'' y^2 + \gamma'' z^2 + \delta'' u^2 = 0,$$

we may take a cubic

$$Ax^3 + By^3 + Cz^3 + Dw^3 = 0,$$

so that the polar conics of three points $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$ of the cubic shall be identical with the conics; as we shall have only 9 conditions to satisfy, and 9 ratios $A : B : C : D$; $x_1 : y_1 : z_1$; $x_2 : y_2 : z_2$; $x_3 : y_3 : z_3$; at command.

We see then that if the conics

$$(a, b, c, f, g, h)(x, y, z)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$(a', b', c', f'\ \ g', h')(x, y, z)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$(a'', b'', c'', f'', g'', h'')(x, y, z)^2 = 0 \ldots\ldots\ldots\ldots (3),$$

have a common self-conjugate quadrilateral they must be polar conics of the same cubic for points $x_1y_1z_1$, x_2y_2z', $x_3y_3z_3$, say.

The polar of (3) for $x_2y_2z_2$ must be the same as the polar of (2) for $x_3y_3z_3$. Expressing the conditions for this, there result three equations, which, together with the two similar triplets derived from the consideration of the other two pairs of conics, will give 9 equations, sufficient for eliminating the 9 quantities $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$.

The three conics must therefore be linked by the condition

$$0 = \begin{vmatrix} \cdot, & \cdot, & \cdot, & -a'', & -h'', & -g'', & a', & k', & g' \\ \cdot, & \cdot, & \cdot, & -h'', & -b'', & -f'', & h', & b', & f' \\ \cdot, & \cdot, & \cdot, & -g'', & -f'', & -c'', & g', & f', & c' \\ a'', & h'', & g'', & \cdot, & \cdot, & \cdot, & -a, & -h, & -g \\ h'', & b'', & f'', & \cdot, & \cdot, & \cdot, & -h, & -b, & -f \\ g'', & f'', & c'', & \cdot, & \cdot, & \cdot, & -g_1, & -f, & -c \\ -a', & -h', & -g', & a, & h, & g, & \cdot, & \cdot, & \cdot \\ -h', & -b', & -f', & h, & b, & f, & \cdot, & \cdot, & \cdot \\ -g', & -f', & -c', & g, & f, & c, & \cdot, & \cdot, & \cdot \end{vmatrix},$$

the right-hand member of which being a skew symmetrical determinant of odd order vanishes identically. It is seen then that any three conics may be polar conics of the same cubic and may have a common self-conjugate quadrilateral.

2.[*] In the same way it may be shewn that two plane cubics having the same self-conjugate pentagon are polar cubics of the same quartic, and their coefficients must satisfy the relation

$$0 = \begin{vmatrix} a, & a_2, & a_3, & a', & a_2', & a_3' \\ b_1, & b, & b_3, & b_1', & b', & b_3' \\ c_1, & c_2, & c, & c_1', & c_2', & c' \\ m, & b_3, & c_2, & m', & b_3', & c_2' \\ a_3, & m, & c_1, & a_3', & m', & c_1' \\ a_2, & b_1, & m, & a_2', & b_1', & m' \end{vmatrix},$$

shewing that the result obtained from counting constants only requires correction; the equations to the cubic being written as in *Salmon*, p. 189.

II. If three solid quadrics have a common self-conjugate 5-hedron we may write them

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta u^2 + \varepsilon w^2 = 0,$$

$$\alpha' x^2 + \beta' y^2 + \gamma' z^2 + \delta' u^2 + \varepsilon' w^2 = 0,$$

$$\alpha'' x^2 + \beta'' y^2 + \gamma'' z^2 + \delta'' u^2 + \varepsilon'' w^2 = 0,$$

and these will be polar quadrics of the cubic

$$Ax^3 + By^3 + Cz^3 + Du^3 + Ew^3 = 0,$$

for the points $x_1 y_1 z_1 w_1$, $x_2 y_2 z_2 w_2$, $x_3 y_3 z_3 w_3$,

if
$$Ax_1/\alpha = By_1/\beta = Cz_1/\gamma = Du_1/\delta = Ew_1/\varepsilon,$$

$$Ax_2/\alpha' = By_2/\beta' = Cz_2/\gamma' = Du_2/\delta' = Ew_2/\varepsilon',$$

$$Ax_3/\alpha'' = By_2/\beta'' = Cz_3/\gamma'' = Du_3/\delta'' = Ew_3/\varepsilon''',$$

12 equations between the 13 ratios

$$A : B : C : D : E; \quad x_1 : y_1 : z_1 : w_1;$$

$$x_2 : y_2 : z_2 : w_2; \quad x_3 : y_3 : z_3 : w_3.$$

The three quadrics are therefore polar quadrics of the same cubic, and are therefore linked by the conditions given *Salmon*, p. 210.

---

[*] Hence a cubic and its Hessian cannot be polars of the same Lüroth quartic unless the cubic is nodal.

III. It may be shown as before, that any three quinary quadrics have an infinity of common self-conjugate 7-hedra, and are polar quadrics of the same cubic, the resulting condition expressing this last fact being satisfied identically, on account of the odd order of the skew symmetrical determinant.

IV. From the table, three senary quadrics appear to have a common self-conjugate 8-hedron. But as before, the quadrics must be polar quadrics of the same cubic, and the determinant equation expressing this fact, viz.

$$0=\begin{vmatrix} \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & -a''_{11}, & -a''_{12}, & -a''_{13}, & -a''_{14}, & -a''_{15}, & -a''_{16}, & a'_{11}, & a'_{12}, & a'_{13}, & a'_{14}, & a'_{15}, & a'_{16} \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & -a''_{21}, & -a''_{22}, & -a''_{23}, & -a''_{24}, & -a''_{25}, & -a''_{26}, & a'_{21}, & a'_{22}, & a'_{23}, & a'_{24}, & a'_{25}, & a'_{26} \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & -a''_{31}, & -a''_{32}, & -a''_{33}, & -a''_{34}, & -a''_{35}, & -a''_{36}, & a'_{31}, & a'_{32}, & a'_{33}, & a'_{34}, & a'_{35}, & a'_{36} \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & -a''_{41}, & -a''_{42}, & -a''_{43}, & -a''_{44}, & -a''_{45}, & -a''_{46}, & a'_{41}, & a'_{42}, & a'_{43}, & a'_{44}, & a'_{45}, & a'_{46} \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & -a''_{51}, & -a''_{52}, & -a''_{53}, & -a''_{54}, & -a''_{55}, & -a''_{56}, & a'_{51}, & a'_{52}, & a'_{53}, & a'_{54}, & a'_{55}, & a'_{56} \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & -a''_{61}, & -a''_{62}, & -a''_{63}, & -a''_{64}, & -a''_{65}, & -a''_{66}, & a'_{61}, & a'_{62}, & a'_{63}, & a'_{64}, & a'_{65}, & a'_{66} \\ a''_{11}, & a''_{12}, & a''_{13}, & a''_{14}, & a''_{15}, & a''_{16}, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & -a_{11}, & -a_{12}, & -a_{13}, & -a_{14}, & -a_{15}, & -a_{16} \\ & & \&c. & & & & & & \&c. & & & & & & \&c. & & & \end{vmatrix}$$

reduces to one of the third order in the coefficients of each quadric, the equation to the quadrics being

$$0=\Sigma a_{11}x^2+2\Sigma a_{12}xy+\&c.$$

And in general, although three $(n+1)$-ary quadrics can have a common self-conjugate $(3n+1)$-hedron, and be polar quadrics of the same cubic, yet three $2n$-ary quadrics cannot have a common self-conjugate $(3n-1)$-hedron, unless they are linked by an equation of the nth order in the coefficients of each of the quadrics.

FORMULÆ IN ELLIPTIC FUNCTIONS.

By *J. W. L. Glaisher.*

§1. The systems of formulæ contained in this paper are of an elementary character, but they seem sufficiently curious to be deserving of notice, if only on account of the form of the expressions involved.

I use throughout the twelve elliptic functions $\operatorname{sn} x$, $\operatorname{cn} x, \ldots,$ $\operatorname{sd} x$, $\operatorname{nd} x$, and in all cases the complete systems of equations are given. I believe that a great part of what is most characteristic in Elliptic Functions is lost by attempting to express all the results by means of only one of the four groups of functions, and that it is important to treat the subject as one having reference to a system of twelve doubly periodic functions which are divisible into four groups, and are such that from any formula relating to one of the functions we may deduce eleven others which with it form a complete system.*

Using an accent to denote differentiation with respect to x, we start with the fundamental formulæ:†

$$\begin{aligned}
(\operatorname{sn} x)' &= \operatorname{cn} x \operatorname{dn} x,\\
(\operatorname{cn} x)' &= -\operatorname{dn} x \operatorname{sn} x,\\
(\operatorname{dn} x)' &= -k^2 \operatorname{sn} x \operatorname{cn} x;
\end{aligned}$$

$$\begin{aligned}
(\operatorname{ns} x)' &= -\operatorname{ds} x \operatorname{cs} x,\\
(\operatorname{ds} x)' &= -\operatorname{cs} x \operatorname{ns} x,\\
(\operatorname{cs} x)' &= -\operatorname{ns} x \operatorname{ds} x;
\end{aligned}$$

$$\begin{aligned}
(\operatorname{dc} x)' &= k'^2 \operatorname{nc} x \operatorname{sc} x,\\
(\operatorname{nc} x)' &= \operatorname{sc} x \operatorname{dc} x,\\
(\operatorname{sc} x)' &= \operatorname{dc} x \operatorname{nc} x;
\end{aligned}$$

$$\begin{aligned}
(\operatorname{cd} x)' &= -k'^2 \operatorname{sd} x \operatorname{nd} x,\\
(\operatorname{sd} x)' &= \operatorname{nd} x \operatorname{cd} x,\\
(\operatorname{nd} x)' &= k^2 \operatorname{cd} x \operatorname{sd} x;
\end{aligned}$$

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[*] *Messenger*, vol. XI. pp. 81–95; 120–138.

[†] *Id.* vol. XI. p. 92.

Derivatives of $\log \operatorname{sn} x$, $\log \operatorname{cn} x$, &c., §§ 2–5.

§ 2. From the above formulæ we immediately derive the following:

$$
\begin{aligned}
(\log \operatorname{sn} x)' &= \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x}, \\
(\log \operatorname{cn} x)' &= -\frac{\operatorname{dn} x \operatorname{sn} x}{\operatorname{cn} x}, \\
(\log \operatorname{dn} x)' &= -k^2 \frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x};
\end{aligned}
$$

$$
\begin{aligned}
(\log \operatorname{ns} x)' &= -\frac{\operatorname{ds} x \operatorname{cs} x}{\operatorname{ns} x}, \\
(\log \operatorname{ds} x)' &= -\frac{\operatorname{cs} x \operatorname{ns} x}{\operatorname{ds} x}, \\
(\log \operatorname{cs} x)' &= -\frac{\operatorname{ns} x \operatorname{ds} x}{\operatorname{cs} x};
\end{aligned}
$$

$$
\begin{aligned}
(\log \operatorname{dc} x)' &= k'^2 \frac{\operatorname{nc} x \operatorname{sc} x}{\operatorname{dc} x}, \\
(\log \operatorname{nc} x)' &= \frac{\operatorname{sc} x \operatorname{dc} x}{\operatorname{nc} x}, \\
(\log \operatorname{sc} x)' &= \frac{\operatorname{dc} x \operatorname{nc} x}{\operatorname{sc} x};
\end{aligned}
$$

$$
\begin{aligned}
(\log \operatorname{cd} x)' &= -k'^2 \frac{\operatorname{sd} x \operatorname{nd} x}{\operatorname{cd} x}, \\
(\log \operatorname{sd} x)' &= \frac{\operatorname{nd} x \operatorname{cd} x}{\operatorname{sd} x}, \\
(\log \operatorname{nd} x)' &= k^2 \frac{\operatorname{cd} x \operatorname{sd} x}{\operatorname{nd} x}.
\end{aligned}
$$

§ 3. Both the squares of these formulæ and their derivatives may be exhibited in rather curious forms, in each of which the function considered and its reciprocal are involved;

viz. for the squares we find

$$\begin{aligned}
\{(\log \operatorname{sn} x)'\}^2 &= k^2 \operatorname{sn}^2 x + \operatorname{ns}^2 x - 1 - k^2, \\
\{(\log \operatorname{cn} x)'\}^2 &= -k^2 \operatorname{cn}^2 x + k'^2 \operatorname{nc}^2 x + k^2 - k'^2, \\
\{(\log \operatorname{dn} x)'\}^2 &= -\operatorname{dn}^2 x - k'^2 \operatorname{nd}^2 x + 1 + k'^2;
\end{aligned}$$

$$\begin{aligned}
\{(\log \operatorname{ns} x)'\}^2 &= \operatorname{ns}^2 x + k^2 \operatorname{sn}^2 x - 1 - k^2, \\
\{(\log \operatorname{ds} x)'\}^2 &= \operatorname{ds}^2 x - k^2 k'^2 \operatorname{sd}^2 x + k^2 - k'^2, \\
\{(\log \operatorname{cs} x)'\}^2 &= \operatorname{cs}^2 x + k'^2 \operatorname{sc}^2 x + 1 + k'^2;
\end{aligned}$$

$$\begin{aligned}
\{(\log \operatorname{dc} x)'\}^2 &= \operatorname{dc}^2 x + k^2 \operatorname{cd}^2 x - 1 - k^2, \\
\{(\log \operatorname{nc} x)'\}^2 &= k'^2 \operatorname{nc}^2 x - k^2 \operatorname{cn}^2 x + k^2 - k'^2, \\
\{(\log \operatorname{sc} x)'\}^2 &= k'^2 \operatorname{sc}^2 x + \operatorname{cs}^2 x + 1 + k'^2;
\end{aligned}$$

$$\begin{aligned}
\{(\log \operatorname{cd} x)'\}^2 &= k^2 \operatorname{cd}^2 x + \operatorname{dc}^2 x - 1 - k^2, \\
\{(\log \operatorname{sd} x)'\}^2 &= -k^2 k'^2 \operatorname{sd}^2 x + \operatorname{ds}^2 x + k^2 - k'^2, \\
\{(\log \operatorname{nd} x)'\}^2 &= -k'^2 \operatorname{nd}^2 x - \operatorname{dn}^2 x + 1 + k'^2;
\end{aligned}$$

and, for the derivatives,

$$\begin{aligned}
(\log \operatorname{sn} x)'' &= k^2 \operatorname{sn}^2 x - \operatorname{ns}^2 x, \\
(\log \operatorname{cn} x)'' &= -k^2 \operatorname{cn}^2 x - k'^2 \operatorname{nc}^2 x, \\
(\log \operatorname{dn} x)'' &= -\operatorname{dn}^2 x + k'^2 \operatorname{nd}^2 x;
\end{aligned}$$

$$\begin{aligned}
(\log \operatorname{ns} x)'' &= \operatorname{ns}^2 x - k^2 \operatorname{sn}^2 x, \\
(\log \operatorname{ds} x)'' &= \operatorname{ds}^2 x + k^2 k'^2 \operatorname{sd}^2 x, \\
(\log \operatorname{cs} x)'' &= \operatorname{cs}^2 x - k'^2 \operatorname{sc}^2 x;
\end{aligned}$$

$$\begin{aligned}
(\log \operatorname{dc} x)'' &= \operatorname{dc}^2 x - k^2 \operatorname{cd}^2 x, \\
(\log \operatorname{nc} x)'' &= k'^2 \operatorname{nc}^2 x + k^2 \operatorname{cn}^2 x, \\
(\log \operatorname{sc} x)'' &= k'^2 \operatorname{sc}^2 x - \operatorname{cs}^2 x;
\end{aligned}$$

$$\begin{aligned}
(\log \operatorname{cd} x)'' &= k^2 \operatorname{cd}^2 x - \operatorname{dc}^2 x, \\
(\log \operatorname{sd} x)'' &= -k^2 k'^2 \operatorname{sd}^2 x - \operatorname{ds}^2 x, \\
(\log \operatorname{nd} x)'' &= -k'^2 \operatorname{nd}^2 x + \operatorname{dn}^2 x.
\end{aligned}$$

§ 4. By differentiating the first group of formulæ in the last section, we find

$$\begin{aligned}
(\log \operatorname{sn} x)'(\log \operatorname{sn} x)'' &= k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x - \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x,\\
(\log \operatorname{cn} x)'(\log \operatorname{cn} x)'' &= k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x + k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x,\\
(\log \operatorname{dn} x)'(\log \operatorname{dn} x)'' &= k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x - k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x;
\end{aligned}$$

$$\begin{aligned}
(\log \operatorname{ns} x)'(\log \operatorname{ns} x)'' &= -\operatorname{ns} x \operatorname{ds} x \operatorname{cs} x + k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x,\\
(\log \operatorname{ds} x)'(\log \operatorname{ds} x)'' &= -\operatorname{ns} x \operatorname{ds} x \operatorname{cs} x - k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x,\\
(\log \operatorname{cs} x)'(\log \operatorname{cs} x)'' &= -\operatorname{ns} x \operatorname{ds} x \operatorname{cs} x + k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x;
\end{aligned}$$

$$\begin{aligned}
(\log \operatorname{dc} x)'(\log \operatorname{dc} x)'' &= k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x - k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x,\\
(\log \operatorname{nc} x)'(\log \operatorname{nc} x)'' &= k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x + k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x,\\
(\log \operatorname{sc} x)'(\log \operatorname{sc} x)'' &= k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x - \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x;
\end{aligned}$$

$$\begin{aligned}
(\log \operatorname{cd} x)'(\log \operatorname{cd} x)'' &= -k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x + k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x,\\
(\log \operatorname{sd} x)'(\log \operatorname{sd} x)'' &= -k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x - \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x,\\
(\log \operatorname{nd} x)'(\log \operatorname{nd} x)'' &= -k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x + k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x;
\end{aligned}$$

and, by differentiating the second group,

$$\begin{aligned}
\tfrac{1}{2}(\log \operatorname{sn} x)''' &= k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x + \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x,\\
\tfrac{1}{2}(\log \operatorname{cn} x)''' &= k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x - k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x,\\
\tfrac{1}{2}(\log \operatorname{dn} x)''' &= k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x + k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x;
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(\log \operatorname{ns} x)''' &= -\operatorname{ns} x \operatorname{ds} x \operatorname{cs} x - k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x,\\
\tfrac{1}{2}(\log \operatorname{ds} x)''' &= -\operatorname{ns} x \operatorname{ds} x \operatorname{cs} x + k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x,\\
\tfrac{1}{2}(\log \operatorname{cs} x)''' &= -\operatorname{ns} x \operatorname{ds} x \operatorname{cs} x - k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x;
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(\log \operatorname{dc} x)''' &= k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x + k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x,\\
\tfrac{1}{2}(\log \operatorname{nc} x)''' &= k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x - k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x,\\
\tfrac{1}{2}(\log \operatorname{sc} x)''' &= k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x + \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x;
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(\log \operatorname{cd} x)''' &= -k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x - k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x,\\
\tfrac{1}{2}(\log \operatorname{sd} x)''' &= -k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x + \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x,\\
\tfrac{1}{2}(\log \operatorname{nd} x)''' &= -k^2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x - k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x.
\end{aligned}$$

The right-hand members of these equations consist of all the combinations that can be formed, by addition and subtraction, from any two of the four quantities

$$\begin{gathered} k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x, \\ \operatorname{ns} x \operatorname{ds} x \ \operatorname{cs} x, \\ k'^2 \operatorname{dc} x \operatorname{nc} x \ \operatorname{sc} x, \\ k^2 k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x. \end{gathered}$$

§ 5. The formulæ contained in the last two sections are those which seemed to me especially noticeable in connexion with the derivatives of $\log \operatorname{sn} x$, &c.: the following forms of the expressions for $\{(\log \operatorname{sn} x)'\}^2$, &c. may, however, also be noticed:

$$\begin{aligned} \{(\log \operatorname{sn} x)'\}^2 &= \operatorname{cs}^2 x - k^2 \operatorname{cn}^2 x &&= \operatorname{ds}^2 x - \operatorname{dn}^2 x, \\ \{(\log \operatorname{cn} x)'\}^2 &= \operatorname{dc}^2 x - \operatorname{dn}^2 x &&= k'^2 \operatorname{sc}^2 x + k^2 \operatorname{sn}^2 x, \\ \{(\log \operatorname{dn} x)'\}^2 &= - k^2 k'^2 \operatorname{sd}^2 x + k^2 \operatorname{sn}^2 x &&= k^2 \operatorname{cd}^2 x - k^2 \operatorname{cn}^2 x; \end{aligned}$$

$$\begin{aligned} \{(\log \operatorname{ns} x)'\}^2 &= - \operatorname{dn}^2 x + \operatorname{ds}^2 x &&= - k^2 \operatorname{cn}^2 x + \operatorname{cs}^2 x, \\ \{(\log \operatorname{ds} x)'\}^2 &= k^2 \operatorname{cd}^2 x + \operatorname{cs}^2 x &&= - k'^2 \operatorname{nd}^2 x + \operatorname{ns}^2 x, \\ \{(\log \operatorname{cs} x)'\}^2 &= k'^2 \operatorname{nc}^2 x + \operatorname{ns}^2 x &&= \operatorname{dc}^2 x + \operatorname{ds}^2 x; \end{aligned}$$

$$\begin{aligned} \{(\log \operatorname{dc} x)'\}^2 &= - k'^2 \operatorname{nd}^2 x + k'^2 \operatorname{nc}^2 x &&= - k^2 k'^2 \operatorname{sd}^2 x + k'^2 \operatorname{sc}^2 x, \\ \{(\log \operatorname{nc} x)'\}^2 &= k^2 \operatorname{sn}^2 x + k'^2 \operatorname{sc}^2 x &&= - \operatorname{dn}^2 x + \operatorname{dc}^2 x, \\ \{(\log \operatorname{sc} x)'\}^2 &= \operatorname{ds}^2 x + \operatorname{dc}^2 x &&= \operatorname{ns}^2 x + k'^2 \operatorname{nc}^2 x; \end{aligned}$$

$$\begin{aligned} \{(\log \operatorname{cd} x)'\}^2 &= k'^2 \operatorname{sc}^2 x - k^2 k'^2 \operatorname{sd}^2 x &&= k'^2 \operatorname{nc}^2 x - k'^2 \operatorname{nd}^2 x, \\ \{(\log \operatorname{sd} x)'\}^2 &= \operatorname{ns}^2 x - k'^2 \operatorname{nd}^2 x &&= \operatorname{cs}^2 x + k^2 \operatorname{cd}^2 x, \\ \{(\log \operatorname{nd} x)'\}^2 &= - k^2 \operatorname{cn}^2 x + k^2 \operatorname{cd}^2 x &&= k^2 \operatorname{sn}^2 x - k^2 k'^2 \operatorname{sd}^2 x. \end{aligned}$$

The second derivatives of $\log \operatorname{sn} x$, &c. may also be expressed as follows:

$$\begin{aligned} (\log \operatorname{sn} x)'' &= \frac{1}{k'^2} (k^2 \operatorname{cs}^2 x \operatorname{cn}^2 x - \operatorname{ds}^2 x \operatorname{dn}^2 x), \\ (\log \operatorname{cn} x)'' &= - \operatorname{dc}^2 x \operatorname{dn}^2 x - k^2 k'^2 \operatorname{sc}^2 x \operatorname{sn}^2 x, \\ (\log \operatorname{dn} x)'' &= k^2 k'^2 \operatorname{sd}^2 x \operatorname{sn}^2 x - k^2 \operatorname{cd}^2 x \operatorname{cn}^2 x; \end{aligned}$$

$$(\log \operatorname{ns} x)'' = \frac{1}{k'^2}(\operatorname{dn}^2 x \operatorname{ds}^2 x - k^2 \operatorname{cn}^2 x \operatorname{cs}^2 x),$$

$$(\log \operatorname{ds} x)'' = k^2 \operatorname{cd}^2 x \operatorname{cs}^2 x + k'^2 \operatorname{nd}^2 x \operatorname{ns}^2 x,$$

$$(\log \operatorname{cs} x)'' = \frac{1}{k^2}(-k'^2 \operatorname{nc}^2 x \operatorname{ns}^2 x + \operatorname{dc}^2 x \operatorname{ds}^2 x);$$

$$(\log \operatorname{dc} x)'' = k'^2 \operatorname{nd}^2 x \operatorname{nc}^2 x - k^2 k'^2 \operatorname{sd}^2 x \operatorname{sc}^2 x,$$

$$(\log \operatorname{nc} x)'' = k^2 k'^2 \operatorname{sn}^2 x \operatorname{sc}^2 x + \operatorname{dn}^2 x \operatorname{dc}^2 x,$$

$$(\log \operatorname{sc} x)'' = \frac{1}{k^2}(-\operatorname{ds}^2 x \operatorname{dc}^2 x + k'^2 \operatorname{ns}^2 x \operatorname{nc}^2 x);$$

$$(\log \operatorname{cd} x)'' = k^2 k'^2 \operatorname{sc}^2 x \operatorname{sd}^2 x - k'^2 \operatorname{nc}^2 x \operatorname{nd}^2 x,$$

$$(\log \operatorname{sd} x)'' = -k'^2 \operatorname{ns}^2 x \operatorname{nd}^2 x - k^2 \operatorname{cs}^2 x \operatorname{cd}^2 x,$$

$$(\log \operatorname{nd} x)'' = k^2 \operatorname{cn}^2 x \operatorname{cd}^2 x - k^2 k'^2 \operatorname{sn}^2 x \operatorname{sd}^2 x.$$

Expressions for $\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{sn} x$, &c., §§ 6–9.

§ 6. Taking the logarithm of each side of the equation

$$\frac{d}{dx}\log \operatorname{sn} x = \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x},$$

we have

$$\log\frac{d}{dx}\log \operatorname{sn} x = \log \operatorname{cn} x + \log \operatorname{dn} x - \log \operatorname{sn} x;$$

and by differentiating the right-hand member of this equation we obtain the expression

$$-\frac{\operatorname{dn} x \operatorname{sn} x}{\operatorname{cn} x} - k^2\frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x} - \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x}$$

$$= -\frac{1 - k^2 \operatorname{sn}^4 x}{\operatorname{sn} x \operatorname{cn} x \operatorname{dn} x}$$

$$= -\frac{2}{\operatorname{sn} 2x}.$$

Thus

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{sn} x = -2\operatorname{ns} 2x.$$

Similarly it can be shown that

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{cn} x = 2\operatorname{ds} 2x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{dn} x = 2\operatorname{cs} 2x.$$

From these formulæ we may deduce at once by the substitution of $x+K$, $x+iK$, $x+K+iK'$ for x the corresponding formulæ for the other groups of functions, which are as follows:

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{ns} x = -2\operatorname{ns} 2x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{ds} x = -2\operatorname{ds} 2x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{cs} x = -2\operatorname{cs} 2x;$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{dc} x = \quad 2\operatorname{ns} 2x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{nc} x = \quad 2\operatorname{ds} 2x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{sc} x = -2\operatorname{cs} 2x;$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{cd} x = \quad 2\operatorname{ns} 2x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{sd} x = -2\operatorname{ds} 2x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\log \operatorname{nd} x = \quad 2\operatorname{cs} 2x.$$

§ 7. If we use λ to denote the operation of logarithmic differentiation, *i.e.* the double operation of taking the logarithm, and then differentiating, so that

$$\lambda = \frac{d}{dx} \log,$$

then the results found in the last section may be written:

$$-\lambda^2 \,\mathrm{sn}\, x = -\lambda^2 \,\mathrm{ns}\, x = \lambda^2 \,\mathrm{dc}\, x = \lambda^2 \,\mathrm{cd}\, x = 2 \,\mathrm{ns}\, 2x,$$

$$\lambda^2 \,\mathrm{cn}\, x = -\lambda^2 \,\mathrm{ds}\, x = \lambda^2 \,\mathrm{nc}\, x = -\lambda^2 \,\mathrm{sd}\, x = 2 \,\mathrm{ds}\, 2x,$$

$$\lambda^2 \,\mathrm{dn}\, x = -\lambda^2 \,\mathrm{cs}\, x = -\lambda^2 \,\mathrm{sc}\, x = \lambda^2 \,\mathrm{nd}\, x = 2 \,\mathrm{cs}\, 2x;$$

from which we deduce at once the following general formulæ, which are true when $n > 1$:

$$\lambda^{2n} \,\mathrm{sn}\, x = \lambda^{2n} \,\mathrm{ns}\, x = \lambda^{2n} \,\mathrm{dc}\, x = \lambda^{2n} \,\mathrm{cd}\, x = -2^n \,\mathrm{ns}\,(2^n x),$$

$$\lambda^{2n} \,\mathrm{cn}\, x = \lambda^{2n} \,\mathrm{ds}\, x = \lambda^{2n} \,\mathrm{nc}\, x = \lambda^{2n} \,\mathrm{sd}\, x = -2^n \,\mathrm{ds}\,(2^n x),$$

$$\lambda^{2n} \,\mathrm{dn}\, x = \lambda^{2n} \,\mathrm{cs}\, x = \lambda^{2n} \,\mathrm{sc}\, x = \lambda^{2n} \,\mathrm{nd}\, x = -2^n \,\mathrm{cs}\,(2^n x).$$

§ 8. The functions $\mathrm{ns}\, x$, $\mathrm{ds}\, x$, $\mathrm{cs}\, x$ thus possess the very singular property of reproducing themselves after an even number of logarithmic differentiations, the exact formulæ being

$$\left(\frac{d}{dx} \log\right)^{2n} \mathrm{ns}\, x = -\, 2^n \,\mathrm{ns}\,(2^n x),$$

$$\left(\frac{d}{dx} \log\right)^{2n} \mathrm{ds}\, x = -\, 2^n \,\mathrm{ds}\,(2^n x),$$

$$\left(\frac{d}{dx} \log\right)^{2n} \mathrm{cs}\, x = -\, 2^n \,\mathrm{cs}\,(2^n x).$$

§ 9. The operation converse to $\dfrac{d}{dx}$ is $\int dx$ and that converse to log is expressed by making the quantity operated upon the exponent of e, so that if

$$\frac{d}{dx} \log \frac{d}{dx} \log u = v,$$

then
$$u = A e^{B \int e^{\int v\, dx}},$$

where A and B are arbitrary constants.

Using $\exp u$ to denote e^u, we thus see that if the constants of integration be properly chosen we must have

$$\exp\int dx \exp\left(-\int dx \operatorname{ns} x\right) = \operatorname{ns} \tfrac{1}{2}x,$$
$$\exp\int dx \exp\left(-\int dx \operatorname{ds} x\right) = \operatorname{ds} \tfrac{1}{2}x,$$
$$\exp\int dx \exp\left(-\int dx \operatorname{cs} x\right) = \operatorname{cs} \tfrac{1}{2}x.$$

These equations may be readily verified by means of the formulæ*

$$-\int dx \operatorname{ns} x = \log(\operatorname{ds} x + \operatorname{cs} x),$$
$$\int dx \operatorname{ds} x = \log(\operatorname{ns} x - \operatorname{cs} x),$$
$$\int dx \operatorname{cs} x = \log(\operatorname{ns} x - \operatorname{ds} x),$$

for, taking as an example the first equation, we have

$$\exp\left(-\int dx \operatorname{ns} x\right) = c\,(\operatorname{ds} x + \operatorname{cs} x)$$

c being the constant of integration. Putting $c = -\frac{1}{2}$ we have

$$\int dx \exp\left(-\int dx \operatorname{ns} x\right) = -\tfrac{1}{2}\log\{(\operatorname{ns} x - \operatorname{cs} x)(\operatorname{ns} x - \operatorname{ds} x)\}$$
$$= -\tfrac{1}{2}\log\left(k^2 \operatorname{sn}^2 \tfrac{1}{2}x\right)$$

which, omitting a constant, $= \log \operatorname{ns} \frac{1}{2}x$.

Values of ns 2*x*, ds 2*x*, cs 2*x*, §§ 10, 11.

§ 10. The expressions for $\operatorname{sn} 2x$, $\operatorname{sd} 2x$, $\operatorname{sc} 2x$ in terms of each of the four groups of functions seem worth giving. They are as follows:

$$\operatorname{ns} 2x = \frac{1 - k^2 \operatorname{sn}^4 x}{2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x} = \frac{\operatorname{ns}^4 x - k^2}{2 \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x}$$
$$= \frac{\operatorname{dc}^4 x - k^2}{2k'^2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x} = \frac{1 - k^2 \operatorname{cd}^4 x}{2k'^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x},$$

$$\operatorname{ds} 2x = \frac{k'^2 + k^2 \operatorname{cn}^4 x}{2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x} = \frac{k^2 k'^2 + \operatorname{ds}^4 x}{2 \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x}$$
$$= \frac{k^2 + k'^2 \operatorname{nc}^4 x}{2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x} = \frac{1 + k^2 k'^2 \operatorname{sd}^4 x}{2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x},$$

$$\operatorname{cs} 2x = \frac{k'^2 - \operatorname{dn}^4 x}{2k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x} = \frac{\operatorname{cs}^4 x - k'^2}{2 \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x}$$
$$= \frac{1 - k'^2 \operatorname{sc}^4 x}{2 \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x} = \frac{1 - k'^2 \operatorname{nd}^4 x}{2k^2 \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x}.$$

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* *Messenger*, vol. XI. p. 94.

§ 11. The four quantities $k^2k'^2\,\mathrm{sn}^4x$, $k^2\,\mathrm{cn}^4x$, dn^4x are connected by the linear relation

$$k'^2 - k^2k'^2\,\mathrm{sn}^4x + k^2\,\mathrm{cn}^4x - \mathrm{dn}^4x = 0,$$

so that each of the numerators $1 - k^2\,\mathrm{sn}^4x$, &c. may be exhibited also in another form as a binomial expression: viz.

$$\begin{aligned} 1 - k^2\,\mathrm{sn}^4x &= \frac{1}{k'^2}\,(\mathrm{dn}^4x - k^2\,\mathrm{cn}^4x), \\ k'^2 + k^2\,\mathrm{cn}^4x &= \mathrm{dn}^4x + k^2k'^2\,\mathrm{sn}^4x, \\ k'^2 - \mathrm{dn}^4x &= k^2k'^2\,\mathrm{sn}^4x - k^2\,\mathrm{cn}^4x. \end{aligned}$$

Similarly from the relations

$$\begin{aligned} k'^2\,\mathrm{ns}^4x - \mathrm{ds}^4x + k^2\,\mathrm{cs}^4x - k^2k'^2 &= 0, \\ \mathrm{dc}^4x - k'^2\,\mathrm{nc}^4x + k^2k'^2\,\mathrm{sc}^4x - k^2 &= 0, \\ k^2\,\mathrm{cd}^4x - k^2k'^2\,\mathrm{sd}^4x + k'^2\,\mathrm{nd}^4x - 1 &= 0, \end{aligned}$$

we deduce alternative forms for the other numerators, viz.

$$\begin{aligned} \mathrm{ns}^4x - k^2 &= \frac{1}{k'^2}\,(\mathrm{ds}^4x - k^2\,\mathrm{cs}^4x), \\ k^2k'^2 + \mathrm{ds}^4x &= k^2\,\mathrm{cs}^4x + k'^2\,\mathrm{ns}^4x, \\ \mathrm{cs}^4x - k'^2 &= \frac{1}{k^2}\,(\mathrm{ds}^4x - k'^2\,\mathrm{ns}^4x); \end{aligned}$$

$$\begin{aligned} \mathrm{dc}^4x - k^2 &= k'^2\,\mathrm{nc}^4x + k^2k'^2\,\mathrm{sc}^4x, \\ k^2 + k'^2\,\mathrm{nc}^4x &= k^2k'^2\,\mathrm{sc}^4x + \mathrm{dc}^4x, \\ 1 - k'^2\,\mathrm{sc}^4x &= \frac{1}{k^2}\,(\mathrm{dc}^4x - k'^2\,\mathrm{nc}^4x); \end{aligned}$$

$$\begin{aligned} 1 - k^2\,\mathrm{cd}^4x &= k'^2\,\mathrm{nd}^4x - k^2k'^2\,\mathrm{sd}^4x, \\ 1 + k^2k'^2\,\mathrm{sd}^4x &= k'^2\,\mathrm{nd}^4x + k^2\,\mathrm{cd}^4x, \\ 1 - k'^2\,\mathrm{nd}^4x &= k^2\,\mathrm{cd}^4x - k^2k'^2\,\mathrm{sd}^4x. \end{aligned}$$

It is unnecessary to write down the formulæ for the other nine functions $\mathrm{sn}\,2x$, $\mathrm{cn}\,2x$, $\mathrm{dn}\,2x$, $\mathrm{dc}\,2x$, &c. as they can be obtained at sight by division from the group $\mathrm{ns}\,2x$, $\mathrm{ds}\,2x$, $\mathrm{cs}\,2x$, and expressed at once in the various binomial forms of which they are susceptible by means of the identical relations just written.

The expressions $\mathrm{cn}^2x\,\mathrm{dn}^2x \pm \mathrm{dn}^2x\,\mathrm{sn}^2x \pm k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x$, &c., §§ 12–15.

§ 12. The three quantities

$$1 - k^2\,\mathrm{sn}^4x = \frac{1}{k'^2}(\mathrm{dn}^4x - k^2\,\mathrm{cn}^4x),$$

$$\frac{1}{k^2}(\mathrm{dn}^4x - k'^2) = \mathrm{cn}^4x - k'^2\,\mathrm{sn}^4x,$$

$$k'^2 + k^2\,\mathrm{cn}^4x = \mathrm{dn}^4x + k^2k'^2\,\mathrm{sn}^4x,$$

when expressed in terms of $\mathrm{sn}\,x$ only are respectively equal to

$$1 - k^2\,\mathrm{sn}^4x,$$

$$1 - 2\,\mathrm{sn}^2x + k^2\,\mathrm{sn}^4x,$$

$$1 - 2k^2\,\mathrm{sn}^2x + k^2\,\mathrm{sn}^4x,$$

and in this form they are the familiar expressions for the common denominator of $\mathrm{sn}\,2x$, $\mathrm{cn}\,2x$, $\mathrm{dn}\,2x$ and for the numerators of the two latter. They may also be expressed respectively in the binomial forms

$$\mathrm{dn}^2x + k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x = \mathrm{cn}^2x + \mathrm{sn}^2x\,\mathrm{dn}^2x,$$

$$\mathrm{cn}^2x - \mathrm{sn}^2x\,\mathrm{dn}^2x = \mathrm{cn}^2x\,\mathrm{dn}^2x - k'^2\,\mathrm{sn}^2x,$$

$$\mathrm{cn}^2x\,\mathrm{dn}^2x + k'^2\,\mathrm{sn}^2x = \mathrm{dn}^2x - k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x,$$

and in the following trinomial forms in which the functions $\mathrm{sn}\,x$, $\mathrm{cn}\,x$, $\mathrm{dn}\,x$ (or, more accurately, the functions $k\,\mathrm{sn}\,x$, $k\,\mathrm{cn}\,x$, $\mathrm{dn}\,x$) are involved symmetrically:

$$\mathrm{cn}^2x\,\mathrm{dn}^2x + \mathrm{dn}^2x\,\mathrm{sn}^2x + k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x,$$

$$\mathrm{cn}^2x\,\mathrm{dn}^2x - \mathrm{dn}^2x\,\mathrm{sn}^2x + k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x,$$

$$\mathrm{cn}^2x\,\mathrm{dn}^2x + \mathrm{dn}^2x\,\mathrm{sn}^2x - k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x;$$

Thus

$$\mathrm{sn}\,2x = \frac{2\,\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}^2x\,\mathrm{dn}^2x + \mathrm{dn}^2x\,\mathrm{sn}^2x + k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x},$$

$$\mathrm{cn}\,2x = \frac{\mathrm{cn}^2x\,\mathrm{dn}^2x - \mathrm{dn}^2x\,\mathrm{sn}^2x + k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x}{\mathrm{cn}^2x\,\mathrm{dn}^2x + \mathrm{dn}^2x\,\mathrm{sn}^2x + k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x},$$

$$\mathrm{dn}\,2x = \frac{\mathrm{cn}^2x\,\mathrm{dn}^2x + \mathrm{dn}^2x\,\mathrm{sn}^2x - k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x}{\mathrm{cn}^2x\,\mathrm{dn}^2x + \mathrm{dn}^2x\,\mathrm{sn}^2x + k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x}.$$

§ 13. Of the four expressions included in

$$\mathrm{cn}^2 x\, \mathrm{dn}^2 x \pm \mathrm{dn}^2 x\, \mathrm{sn}^2 x \pm k^2\, \mathrm{sn}^2 x\, \mathrm{cn}^2 x$$

the one which does not occur in the above equations is

$$\mathrm{cn}^2 x\, \mathrm{dn}^2 x - \mathrm{dn}^2 x\, \mathrm{sn}^2 x - k^2\, \mathrm{sn}^2 x\, \mathrm{cn}^2 x\,;$$

the expression is however noticeable on account of its being the derivative of $\mathrm{sn}\, x\, \mathrm{cn}\, x\, \mathrm{dn}\, x$.

We thus have the formulæ:

$$\mathrm{cn}^2 x\, \mathrm{dn}^2 x + \mathrm{dn}^2 x\, \mathrm{sn}^2 x + k^2\, \mathrm{sn}^2 x\, \mathrm{cn}^2 x = 2\, \mathrm{sn}\, x\, \mathrm{cn}\, x\, \mathrm{dn}\, x\, \mathrm{ns}\, 2x,$$

$$\mathrm{cn}^2 x\, \mathrm{dn}^2 x + \mathrm{dn}^2 x\, \mathrm{sn}^2 x - k^2\, \mathrm{sn}^2 x\, \mathrm{cn}^2 x = 2\, \mathrm{sn}\, x\, \mathrm{cn}\, x\, \mathrm{dn}\, x\, \mathrm{ds}\, 2x,$$

$$\mathrm{cn}^2 x\, \mathrm{dn}^2 x - \mathrm{dn}^2 x\, \mathrm{sn}^2 x + k^2\, \mathrm{sn}^2 x\, \mathrm{cn}^2 x = 2\, \mathrm{sn}\, x\, \mathrm{cn}\, x\, \mathrm{dn}\, x\, \mathrm{cs}\, 2x,$$

$$-\mathrm{cn}^2 x\, \mathrm{dn}^2 x + \mathrm{dn}^2 x\, \mathrm{sn}^2 x + k^2\, \mathrm{sn}^2 x\, \mathrm{cn}^2 x = -\frac{d}{dx}(\mathrm{sn}\, x\, \mathrm{cn}\, x\, \mathrm{dn}\, x)$$

$$= -\tfrac{1}{2}\frac{d^2}{dx^2}\mathrm{sn}^2 x = \tfrac{1}{2}\frac{d^2}{dx^2}\mathrm{cn}^2 x = \tfrac{1}{2}\frac{1}{k^2}\frac{d^2}{dx^2}\mathrm{dn}^2 x.$$

Similarly we find, for the other groups,

$$-\mathrm{ds}^2 x\, \mathrm{cs}^2 x + \mathrm{cs}^2 x\, \mathrm{ns}^2 x + \mathrm{ns}^2 x\, \mathrm{ds}^2 x = 2\, \mathrm{ns}\, x\, \mathrm{ds}\, x\, \mathrm{cs}\, x\, \mathrm{ns}\, 2x,$$

$$\mathrm{ds}^2 x\, \mathrm{cs}^2 x - \mathrm{cs}^2 x\, \mathrm{ns}^2 x + \mathrm{ns}^2 x\, \mathrm{ds}^2 x = 2\, \mathrm{ns}\, x\, \mathrm{ds}\, x\, \mathrm{cs}\, x\, \mathrm{ds}\, 2x,$$

$$\mathrm{ds}^2 x\, \mathrm{cs}^2 x + \mathrm{cs}^2 x\, \mathrm{ns}^2 x - \mathrm{ns}^2 x\, \mathrm{ds}^2 x = 2\, \mathrm{ns}\, x\, \mathrm{ds}\, x\, \mathrm{cs}\, x\, \mathrm{cs}\, 2x,$$

$$\mathrm{ds}^2 x\, \mathrm{cs}^2 x + \mathrm{cs}^2 x\, \mathrm{ns}^2 x + \mathrm{ns}^2 x\, \mathrm{ds}^2 x = -\frac{d}{dx}(\mathrm{ns}\, x\, \mathrm{ds}\, x\, \mathrm{cs}\, x)$$

$$= \tfrac{1}{2}\frac{d^2}{dx^2}\mathrm{ns}^2 x = \tfrac{1}{2}\frac{d^2}{dx^2}\mathrm{cs}^2 x = \tfrac{1}{2}\frac{d^2}{dx^2}\mathrm{ds}^2 x\,;$$

$$-k'^2\, \mathrm{nc}^2 x\, \mathrm{sc}^2 x + \mathrm{sc}^2 x\, \mathrm{dc}^2 x + \mathrm{dc}^2 x\, \mathrm{nc}^2 x = 2\, \mathrm{dc}\, x\, \mathrm{nc}\, x\, \mathrm{sc}\, x\, \mathrm{ns}\, 2x,$$

$$k'^2\, \mathrm{nc}^2 x\, \mathrm{sc}^2 x - \mathrm{sc}^2 x\, \mathrm{dc}^2 x + \mathrm{dc}^2 x\, \mathrm{nc}^2 x = 2\, \mathrm{dc}\, x\, \mathrm{nc}\, x\, \mathrm{sc}\, x\, \mathrm{ds}\, 2x,$$

$$k'^2\, \mathrm{nc}^2 x\, \mathrm{sc}^2 x + \mathrm{sc}^2 x\, \mathrm{dc}^2 x - \mathrm{dc}^2 x\, \mathrm{nc}^2 x = -2\, \mathrm{dc}\, x\, \mathrm{nc}\, x\, \mathrm{sc}\, x\, \mathrm{cs}\, 2x,$$

$$k'^2\, \mathrm{nc}^2 x\, \mathrm{sc}^2 x + \mathrm{sc}^2 x\, \mathrm{dc}^2 x + \mathrm{dc}^2 x\, \mathrm{nc}^2 x = \frac{d}{dx}(\mathrm{dc}\, x\, \mathrm{nc}\, x\, \mathrm{sc}\, x)$$

$$= \tfrac{1}{2}\frac{1}{k'^2}\frac{d^2}{dx^2}\mathrm{dc}^2 x = \tfrac{1}{2}\frac{d^2}{dx^2}\mathrm{nc}^2 x = \tfrac{1}{2}\frac{d^2}{dx^2}\mathrm{sc}^2 x\,;$$

and

$$k'^2\,\mathrm{sd}^2x\,\mathrm{nd}^2x+\mathrm{nd}^2x\,\mathrm{cd}^2x+k^2\,\mathrm{cd}^2x\,\mathrm{sd}^2x=2\,\mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{nd}\,x\,\mathrm{ns}\,2x,$$

$$k'^2\,\mathrm{sd}^2x\,\mathrm{nd}^2x+\mathrm{nd}^2x\,\mathrm{cd}^2x-k^2\,\mathrm{cd}^2x\,\mathrm{sd}^2x=2\,\mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{nd}\,x\,\mathrm{ds}\,2x,$$

$$k'^2\,\mathrm{sd}^2x\,\mathrm{nd}^2x-\mathrm{nd}^2x\,\mathrm{cd}^2x+k^2\,\mathrm{cd}^2x\,\mathrm{sd}^2x=-2\,\mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{nd}\,x\,\mathrm{cs}\,2x,$$

$$-k'^2\,\mathrm{sd}^2x\,\mathrm{nd}^2x+\mathrm{nd}^2x\,\mathrm{cd}^2x+k^2\,\mathrm{cd}^2x\,\mathrm{sd}^2x=\frac{d}{dx}(\mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{nd}\,x)$$

$$=-\tfrac{1}{2}\frac{1}{k'^2}\frac{d^2}{dx^2}\mathrm{cd}^2x=\tfrac{1}{2}\frac{d^2}{dx^2}\mathrm{sd}^2x=\tfrac{1}{2}\frac{1}{k^2}\frac{d^2}{dx^2}\mathrm{nd}^2x.$$

§ 14. The most regular of the systems of formulæ, as is invariably the case, is that relating to the group $\mathrm{ns}\,x$, $\mathrm{ds}\,x$, $\mathrm{cs}\,x$; and, selecting this group, we have

$$\mathrm{ns}\,2x=\frac{-\,\mathrm{ds}^2x\,\mathrm{cs}^2x+\mathrm{cs}^2x\,\mathrm{ns}^2x+\mathrm{ns}^2x\,\mathrm{ds}^2x}{2\,\mathrm{ns}x\,\mathrm{ds}x\,\mathrm{cs}x},$$

$$\mathrm{ds}\,2x=\frac{\mathrm{ds}^2x\,\mathrm{cs}^2x-\mathrm{cs}^2x\,\mathrm{ns}^2x+\mathrm{ns}^2x\,\mathrm{ds}^2x}{2\,\mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{cs}\,x},$$

$$\mathrm{cs}\,2x=\frac{\mathrm{ds}^2x\,\mathrm{cs}^2x+\mathrm{cs}^2x\,\mathrm{ns}^2x-\mathrm{ns}^2x\,\mathrm{ds}^2x}{2\,\mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{cs}\,x}.$$

§ 15. If we put

$$a=\mathrm{ds}^2x\,\mathrm{cs}^2x,\quad b=\mathrm{cs}^2x\,\mathrm{ns}^2x,\quad c=\mathrm{ns}^2x\,\mathrm{ds}^2x,$$

and $s=\tfrac{1}{2}(a+b+c)$, then

$$\mathrm{ns}\,2x=\frac{s-a}{\sqrt[4]{(abc)}},\quad \mathrm{ds}\,2x=\frac{s-b}{\sqrt[4]{(abc)}},\quad \mathrm{cs}\,2x=\frac{s-c}{\sqrt[4]{(abc)}},$$

whence
$$\mathrm{ns}\,2x+\mathrm{ds}\,2x+\mathrm{cs}\,2x=\frac{s}{\sqrt[4]{(abc)}},$$

and also
$$s=-\tfrac{1}{2}\{\sqrt[4]{(abc)}\}'.$$

Thus $s(s-a)(s-b)(s-c)$

$$=\mathrm{ns}^4x\,\mathrm{ds}^4x\,\mathrm{cs}^4x\,\mathrm{ns}\,2x\,\mathrm{ds}\,2x\,\mathrm{cs}\,2x\,(\mathrm{ns}\,2x+\mathrm{ds}\,2x+\mathrm{cs}\,2x)$$

and also

$$=-\tfrac{1}{2}\,\mathrm{ns}\,2x\,\mathrm{ds}\,2x\,\mathrm{cs}\,2x\,\mathrm{ns}^3x\,\mathrm{ds}^3x\,\mathrm{cs}^2x\,\frac{d}{dx}(\mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{cs}\,x),$$

and, equating these expressions to

$$\tfrac{1}{16}\,(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4),$$

we obtain the equalities

$$2\,\mathrm{ns}^4x\,\mathrm{ds}^4x\,\mathrm{cs}^4x\,(\mathrm{ns}^4x+\mathrm{ds}^4x+\mathrm{cs}^4x)-\mathrm{ds}^8x\,\mathrm{cs}^8x-\mathrm{cs}^8x\,\mathrm{ns}^8x-\mathrm{ns}^8x\,\mathrm{ds}^8x$$

$$=16\,\mathrm{ns}^4x\,\mathrm{ds}^4x\,\mathrm{cs}^4x\,\mathrm{ns}\,2x\,\mathrm{ds}\,2x\,\mathrm{cs}\,x\,(\mathrm{ns}\,2x+\mathrm{ds}\,2x+\mathrm{cs}^2 2x)$$

$$=-\,2\,\mathrm{ns}\,2x\,\mathrm{ds}\,2x\,\mathrm{cs}\,2x\,\frac{d}{dx}\,(\mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{cs}\,x)^4.$$

The identity arising from the equality of the first two of these three expressions may be written also in the form

$$16\,\mathrm{ns}\,2x\,\mathrm{ds}\,2x\,\mathrm{cs}\,2x\,(\mathrm{ns}\,2x+\mathrm{ds}\,2x+\mathrm{cs}\,2x)$$

$$=2\,(\mathrm{ns}^4x+\mathrm{ds}^4x+\mathrm{cs}^4x)-\frac{\mathrm{ds}^4x\,\mathrm{cs}^4x}{\mathrm{ns}^4x}-\frac{\mathrm{cs}^4x\,\mathrm{ns}^4x}{\mathrm{ds}^4x}-\frac{\mathrm{ns}^4x\,\mathrm{ds}^4x}{\mathrm{cs}^4x}.$$

Of course many other formulæ of the same kind may be easily obtained.

The expressions $\mathrm{ns}\,x\pm\mathrm{ds}\,x\pm\mathrm{cos}\,x$, &c., §§ 16, 17.

§ 16. From § 13, by combining the first three equations in each group, we find

$$\mathrm{cn}^2x\,\mathrm{dn}^2x-\mathrm{dn}^2x\,\mathrm{sn}^2x-k^2\,\mathrm{sn}^2x\,\mathrm{cn}^2x$$
$$=2\,\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x\,(-\,\mathrm{ns}\,2x+\mathrm{ds}\,2x+\mathrm{cs}\,2x),$$

$$\mathrm{ds}^2x\,\mathrm{cs}^2x+\mathrm{cs}^2x\,\mathrm{ns}^2x+\mathrm{ns}^2x\,\mathrm{ds}^2x$$
$$=2\,\mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{cs}\,x\,(\mathrm{ns}\,2x+\mathrm{ds}\,2x+\mathrm{cs}\,2x),$$

$$k'^2\,\mathrm{nc}^2x\,\mathrm{sc}^2x+\mathrm{sc}^2x\,\mathrm{dc}^2x+\mathrm{dc}^2x\,\mathrm{nc}^2x$$
$$=2\,\mathrm{dc}\,x\,\mathrm{nc}\,x\,\mathrm{sc}\,x\,(\mathrm{ns}\,2x+\mathrm{ds}\,2x-\mathrm{cs}\,2x),$$

$$-\,k'^2\,\mathrm{sd}^2x\,\mathrm{nd}^2x+\mathrm{nd}^2x\,\mathrm{cd}^2x+k^2\,\mathrm{cd}^2x\,\mathrm{sd}^2x$$
$$=2\,\mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{nd}\,x\,(\mathrm{ns}\,2x-\mathrm{ds}\,2x+\mathrm{cs}\,2x),$$

whence

$$\frac{d}{dx}\log\,(\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x)=2\,(-\,\mathrm{ns}\,2x+\mathrm{ds}\,2x+\mathrm{cs}\,2x),$$

$$-\,\frac{d}{dx}\log\,(\mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{cs}\,x)=2\,(\quad\mathrm{ns}\,2x+\mathrm{ds}\,2x+\mathrm{cs}\,2x),$$

$$\frac{d}{dx}\log(\mathrm{dc}\,x\;\mathrm{nc}\,x\;\mathrm{sc}\,x) = 2(\mathrm{ns}\,2x + \mathrm{ds}\,2x - \mathrm{cs}\,2x).$$

$$\frac{d}{dx}\log(\mathrm{cd}\,x\;\mathrm{sd}\,x\;\mathrm{nd}\,x) = 2(\mathrm{ns}\,2x - \mathrm{ds}\,2x + \mathrm{cs}\,2x),$$

or, writing $\frac{1}{2}x$ for x,

$$\frac{d}{dx}\log(\mathrm{sn}\,\tfrac{1}{2}x\;\mathrm{cn}\,\tfrac{1}{2}x\;\mathrm{dn}\,\tfrac{1}{2}x) = -\,\mathrm{ns}\,x + \mathrm{ds}\,x + \mathrm{cs}\,x,$$

$$-\frac{d}{dx}\log(\mathrm{ns}\,\tfrac{1}{2}x\;\mathrm{ds}\,\tfrac{1}{2}x\;\mathrm{cs}\,\tfrac{1}{2}x) = \mathrm{ns}\,x + \mathrm{ds}\,x + \mathrm{cs}\,x,$$

$$\frac{d}{dx}\log(\mathrm{dc}\,\tfrac{1}{2}x\;\mathrm{nc}\,\tfrac{1}{2}x\;\mathrm{sc}\,\tfrac{1}{2}x) = \mathrm{ns}\,x + \mathrm{ds}\,x - \mathrm{cs}\,x,$$

$$\frac{d}{dx}\log(\mathrm{cd}\,\tfrac{1}{2}x\;\mathrm{sd}\,\tfrac{1}{2}x\;\mathrm{nd}\,\tfrac{1}{2}x) = \mathrm{ns}\,x - \mathrm{ds}\,x + \mathrm{cs}\,x.$$

§ 17. These equations may be written also in the slightly different forms:

$$\frac{d}{dx}\log\frac{d}{dx}\,\mathrm{sn}^2\tfrac{1}{2}x = \frac{d}{dx}\log\frac{d}{dx}\,\mathrm{cn}^2\tfrac{1}{2}x = \frac{d}{dx}\log\frac{d}{dx}\,\mathrm{dn}^2\tfrac{1}{2}x$$
$$= -\,\mathrm{ns}\,x + \mathrm{ds}\,x + \mathrm{cs}\,x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\,\mathrm{ns}^2\tfrac{1}{2}x = \frac{d}{dx}\log\frac{d}{dx}\,\mathrm{ds}^2\tfrac{1}{2}x = \frac{d}{dx}\log\frac{d}{dx}\,\mathrm{cs}^2\tfrac{1}{2}x$$
$$= -\,\mathrm{ns}\,x - \mathrm{ds}\,x - \mathrm{cs}\,x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\,\mathrm{dc}^2\tfrac{1}{2}x = \frac{d}{dx}\log\frac{d}{dx}\,\mathrm{nc}^2\tfrac{1}{2}x = \frac{d}{dx}\log\frac{d}{dx}\,\mathrm{sc}^2\tfrac{1}{2}x$$
$$= \mathrm{ns}\,x + \mathrm{ds}\,x - \mathrm{cs}\,x,$$

$$\frac{d}{dx}\log\frac{d}{dx}\,\mathrm{cd}^2\tfrac{1}{2}x = \frac{d}{dx}\log\frac{d}{dx}\,\mathrm{sd}^2\tfrac{1}{2}x = \frac{d}{dx}\log\frac{d}{dx}\,\mathrm{nd}^2\tfrac{1}{2}x$$
$$= \mathrm{ns}\,x - \mathrm{ds}\,x + \mathrm{cs}\,x.$$

The expressions $k'^2\,\mathrm{ns}\,x \pm \mathrm{ds}\,x \pm k^2\,\mathrm{cs}\,x$, &c., §§ 18–20.

§ 18. It can be easily shown that

$$k'^2 + \mathrm{dn}\,2x + k^2\,\mathrm{cn}\,2x = \frac{2\,\mathrm{dn}^4 x}{1 - k^2\,\mathrm{sn}^4 x},$$

$$-k'^2 + \mathrm{dn}\,2x + k^2\,\mathrm{cn}\,2x = \frac{2\,k^2\,\mathrm{cn}^4 x}{1 - k^2\,\mathrm{sn}^4 x},$$

$$k'^2 + \mathrm{dn}\,2x - k^2\,\mathrm{cn}\,2x = \frac{2k'^2}{1 - k^2\,\mathrm{sn}^4 x},$$

$$k'^2 - \mathrm{dn}\,2x + k^2\,\mathrm{cn}\,2x = -\frac{2k^2k'^2}{1 - k^2\,\mathrm{sn}^4 x};$$

whence, dividing by $\mathrm{sn}\,2x$,

$$k'^2\,\mathrm{ns}\,2x + \mathrm{ds}\,2x + k^2\,\mathrm{cs}\,2x = \frac{1}{\mathrm{cd}\,x\ \mathrm{sd}\,x\ \mathrm{nd}\,x},$$

$$-k'^2\,\mathrm{ns}\,2x + \mathrm{ds}\,2x + k^2\,\mathrm{cs}\,2x = \frac{k^2}{\mathrm{dc}\,x\ \mathrm{nc}\,x\ \mathrm{sc}\,x},$$

$$k'^2\,\mathrm{ns}\,2x + \mathrm{ds}\,2x - k^2\,\mathrm{cs}\,2x = \frac{k'^2}{\mathrm{sn}\,x\ \mathrm{cn}\,x\ \mathrm{dn}\,x},$$

$$k'^2\,\mathrm{ns}\,2x - \mathrm{ds}\,2x + k^2\,\mathrm{cs}\,2x = -\frac{k^2k'^2}{\mathrm{ns}\,x\ \ \mathrm{cs}\,x\ \ \mathrm{ds}\,x};$$

or, writing $\tfrac{1}{2}x$ for x, transposing the members of the equations and changing their order,

$$\frac{k'^2}{\mathrm{sn}\,\tfrac{1}{2}x\ \mathrm{cn}\,\tfrac{1}{2}x\ \mathrm{dn}\,\tfrac{1}{2}x} = k'^2\,\mathrm{ns}\,x + \mathrm{ds}\,x - k^2\,\mathrm{cs}\,x,$$

$$-\frac{k^2k'^2}{\mathrm{ns}\,\tfrac{1}{2}x\ \ \mathrm{cs}\,\tfrac{1}{2}x\ \ \mathrm{ds}\,\tfrac{1}{2}x} = k'^2\,\mathrm{ns}\,x - \mathrm{ds}\,x + k^2\,\mathrm{cs}\,x,$$

$$\frac{k^2}{\mathrm{dc}\,\tfrac{1}{2}x\ \ \mathrm{nc}\,\tfrac{1}{2}x\ \ \mathrm{sc}\,\tfrac{1}{2}x} = -k'^2\,\mathrm{ns}\,x + \mathrm{ds}\,x + k^2\,\mathrm{cs}\,x,$$

$$\frac{1}{\mathrm{cd}\,\tfrac{1}{2}x\ \mathrm{sd}\,\tfrac{1}{2}x\ \mathrm{nd}\,\tfrac{1}{2}x} = k'^2\,\mathrm{ns}\,x + \mathrm{ds}\,x + k^2\,\mathrm{cs}\,x.$$

It is interesting to compare these results with those given above, at the end of § 16, to which they bear a general resemblance.*

§ 19. By combining the two groups, we find

$$-\frac{d}{dx}\log(\ \ k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x-k^2\,\mathrm{cs}\,x)=-\,\mathrm{ns}\,x+\mathrm{ds}\,x+\mathrm{cs}\,x,$$

$$\frac{d}{dx}\log(\ \ k'^2\,\mathrm{ns}\,x-\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x)=\ \ \mathrm{ns}\,x+\mathrm{ds}\,x+\mathrm{cs}\,x,$$

$$-\frac{d}{dx}\log(-k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x)=\ \ \mathrm{ns}\,x+\mathrm{ds}\,x-\mathrm{cs}\,x,$$

$$-\frac{d}{dx}\log(\ \ k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x)=\ \ \mathrm{ns}\,x-\mathrm{ds}\,x+\mathrm{cs}\,x.$$

§ 20. By performing the differentiations we obtain the following identities which may be readily verified:

$$\begin{aligned}&k'^2\,\mathrm{ds}\,x\,\mathrm{cs}\,x+\mathrm{cs}\,x\,\mathrm{ns}\,x-k^2\,\mathrm{ns}\,x\,\mathrm{ds}\,x\\&\qquad=(k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x-k^2\,\mathrm{cs}\,x)(-\,\mathrm{ns}\,x+\mathrm{ds}\,x+\mathrm{cs}\,x),\end{aligned}$$

$$\begin{aligned}&k'^2\,\mathrm{ds}\,x\,\mathrm{cs}\,x-\mathrm{cs}\,x\,\mathrm{ns}\,x+k^2\,\mathrm{ns}\,x\,\mathrm{ds}\,x\\&\qquad=-(k'^2\,\mathrm{ns}\,x-\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x)(\mathrm{ns}\,x+\mathrm{ds}\,x+\mathrm{cs}\,x),\end{aligned}$$

$$\begin{aligned}&-k'^2\,\mathrm{ds}\,x\,\mathrm{cs}\,x+\mathrm{cs}\,x\,\mathrm{ns}\,x+k^2\,\mathrm{ns}\,x\,\mathrm{ds}\,x\\&\qquad=(-k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x)(\mathrm{ns}\,x+\mathrm{ds}\,x-\mathrm{cs}\,x),\end{aligned}$$

$$\begin{aligned}&k'^2\,\mathrm{ds}\,x\,\mathrm{cs}\,x+\mathrm{cs}\,x\,\mathrm{ns}\,x+k^2\,\mathrm{ns}\,x\,\mathrm{ds}\,x\\&\qquad=(k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x)(\mathrm{ns}\,x-\mathrm{ds}\,x+\mathrm{cs}\,x).\end{aligned}$$

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* From these groups of formulæ we see that if A, B, C denote respectively

$$1-k^2\,\mathrm{sn}^4x,\qquad 1-2\,\mathrm{sn}^2x+k^2\,\mathrm{sn}^4x,\qquad 1-2k^2\,\mathrm{sn}^2x+k^2\,\mathrm{sn}^4x,$$

and if s, c, d denote $\mathrm{sn}\,\frac{1}{2}x$, $\mathrm{cn}\,\frac{1}{2}x$, $\mathrm{dn}\,\frac{1}{2}x$, then

$$\begin{aligned}-A+B+C&=-scd\frac{d}{dx}\log\frac{1}{scd}, & k'^2A-k^2B+C&=\frac{k'^2}{scd},\\ A+B+C&=\ \ scd\frac{d}{dx}\log\frac{s^3}{cd}, & k'^2A+k^2B-C&=-\frac{k^2k'^2s^3}{cd},\\ A-B+C&=-scd\frac{d}{dx}\log\frac{c^3}{ds}, & -k'^2A+k^2B+C&=\frac{k^2c^3}{ds},\\ A+B-C&=-scd\frac{d}{dx}\log\frac{d^3}{sc}, & k'^2A+k^2B+C&=\frac{d^3}{sc}.\end{aligned}$$

It may be noticed that we have also the identity

$$(k'^2 \operatorname{ns} x + \operatorname{ds} x + k^2 \operatorname{cs} x)(-k'^2 \operatorname{ns} x + \operatorname{ds} x + k^2 \operatorname{cs} x)$$

$$\times (k'^2 \operatorname{ns} x - \operatorname{ds} x + k^2 \operatorname{cs} x)(\ k'^2 \operatorname{ns} x + \operatorname{ds} x - k^2 \operatorname{cs} x) = -k^4 k'^4;$$

or, writing the product in its expanded form,

$$k'^8 \operatorname{ns}^4 x + \operatorname{ds}^4 x + k^8 \operatorname{cs}^4 x - 2k^4 \operatorname{ds}^2 x \operatorname{cs}^2 x - 2k^4 k'^4 \operatorname{cs}^2 x \operatorname{ns}^2 x$$

$$- 2k'^4 \operatorname{ns}^2 x \operatorname{ds}^2 x = k^4 k'^4.$$

Values of $\int \operatorname{ns} x\, dx$, &c., §§ 21, 22.

§ 21. We may write the formulæ giving the integrals of $\operatorname{ns} x$, &c. (§ 9) in the symmetrical form:

$$\int \operatorname{ns} x\, dx = \log(\operatorname{ds} x - \operatorname{cs} x),$$

$$\int \operatorname{ds} x\, dx = \log(\operatorname{cs} x - \operatorname{ns} x),$$

$$\int \operatorname{cs} x\, dx = \log(\operatorname{ns} x - \operatorname{ds} x);$$

and, from § 19, we find

$$\int(\ \operatorname{ns} x + \operatorname{ds} x + \operatorname{cs} x)\, dx = \ \log(\ k'^2 \operatorname{ns} x - \operatorname{ds} x + k^2 \operatorname{cs} x),$$

$$\int(-\operatorname{ns} x + \operatorname{ds} x + \operatorname{cs} x)\, dx = -\log(\ k'^2 \operatorname{ns} x + \operatorname{ds} x - k^2 \operatorname{cs} x),$$

$$\int(\ \operatorname{ns} x - \operatorname{ds} x + \operatorname{cs} x)\, dx = -\log(\ k'^2 \operatorname{ns} x + \operatorname{ds} x + k^2 \operatorname{cs} x),$$

$$\int(\ \operatorname{ns} x + \operatorname{ds} x - \operatorname{cs} x)\, dx = -\log(-k'^2 \operatorname{ns} x + \operatorname{ds} x + k^2 \operatorname{cs} x).$$

We may write these results also in the forms:

$$\int \operatorname{ns} x\, dx = \log \frac{\operatorname{ns} \frac{1}{2}x}{\operatorname{ds} \frac{1}{2}x \operatorname{cs} \frac{1}{2}x},$$

$$\int \operatorname{ds} x\, dx = \log \frac{\operatorname{ds} \frac{1}{2}x}{\operatorname{cs} \frac{1}{2}x \operatorname{ns} \frac{1}{2}x},$$

$$\int \operatorname{cs} x\, dx = \log \frac{\operatorname{cs} \frac{1}{2}x}{\operatorname{ns} \frac{1}{2}x \operatorname{ds} \frac{1}{2}x};$$

$$\int(\ \mathrm{ns}\,x+\mathrm{ds}\,x+\mathrm{cs}\,x)\,dx=-\ \log(\mathrm{ns}\,\tfrac{1}{2}x\ \mathrm{ds}\,\tfrac{1}{2}x\ \ \mathrm{cs}\,\tfrac{1}{2}x),$$

$$\int(-\mathrm{ns}\,x+\mathrm{ds}\,x+\mathrm{cs}\,x)\,dx=\ \log(\mathrm{sn}\,\tfrac{1}{2}x\ \mathrm{cn}\,\tfrac{1}{2}x\ \mathrm{dn}\,\tfrac{1}{2}x),$$

$$\int(\ \mathrm{ns}\,x-\mathrm{ds}\,x+\mathrm{cs}\,x)\,dx=\ \log(\mathrm{cd}\,\tfrac{1}{2}x\ \mathrm{sd}\,\tfrac{1}{2}x\ \mathrm{nd}\,\tfrac{1}{2}x).$$

$$\int(\ \mathrm{ns}\,x+\mathrm{ds}\,x-\mathrm{cs}\,x)\,dx=\ \log(\mathrm{dc}\,\tfrac{1}{2}x\ \mathrm{nc}\,\tfrac{1}{2}x\ \ \mathrm{sc}\,\tfrac{1}{2}x).$$

It is easy to verify that these seven formulæ are consistent with one another.

§22. By comparing the first two groups of formulæ in the last section, we are led to the equalities;

$$k'^2\,\mathrm{ns}\,x-\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x=\quad(\mathrm{ds}\,x-\mathrm{cs}\,x)(\mathrm{cs}\,x-\mathrm{ns}\,x)(\mathrm{ns}\,x-\mathrm{ds}\,x),$$

$$k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x-k^2\,\mathrm{cs}\,x=-k^2\,\frac{\mathrm{ds}\,x-\mathrm{cs}\,x}{(\mathrm{cs}\,x-\mathrm{ns}\,x)\,(\mathrm{ns}\,x-\mathrm{ds}\,x)},$$

$$k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x=-k^2k'^2\,\frac{\mathrm{cs}\,x-\mathrm{ns}\,x}{(\mathrm{ns}\,x-\mathrm{ds}\,x)\,(\mathrm{ds}\,x-\mathrm{cs}\,x)},$$

$$-k'^2\,\mathrm{ns}\,x+\mathrm{ds}\,x+k^2\,\mathrm{cs}\,x=-k'^2\,\frac{\mathrm{ns}\,x-\mathrm{ds}\,x}{(\mathrm{ds}\,x-\mathrm{cs}\,x)\,(\mathrm{cs}\,x-\mathrm{ns}\,x)}.$$

Remarks on the twelve functions, § 23.

§23. I may remark in conclusion that perhaps the most convenient forms of the twelve functions are:

$$\begin{array}{ccc}
-k\,\mathrm{sn}\,x, & ik\,\mathrm{cn}\,x, & i\,\mathrm{dn}\,x\,; \\
-\mathrm{ns}\,x, & -\mathrm{ds}\,x, & -\mathrm{cs}\,x\,; \\
\mathrm{dc}\,x, & k'\,\mathrm{nc}\,x, & k'\,\mathrm{sc}\,x\,; \\
k\,\mathrm{cd}\,x, & ikk'\,\mathrm{sd}\,x, & ik'\,\mathrm{nd}\,x.
\end{array}$$

The ns, ds, cs functions are affected only with the negative sign (not with i), and the twelve functions, in this form, are such that the derivative of each function is equal to the simple product of the other two members of the same group, *i.e.* for example,

$$(-k\,\mathrm{sn}\,x)'=(ik\,\mathrm{cn}\,x)\times(i\,\mathrm{dn}\,x),$$

$$(ik\,\mathrm{cn}\,x)'=(i\,\mathrm{dn}\,x)\times(-k\,\mathrm{sn}\,x),$$

$$(-\mathrm{ns}\,x)'=(-\mathrm{ds}\,x)\times(-\mathrm{cs}\,x),$$

&c. &c.

The other three groups are deducible from the group dc$x$, k' nc$x$, k' sc$x$ by the substitution of $x+K$, $x+iK'$, $x+K+iK'$ for x.

The ns, ds, cs group differs from the other groups by the fact that the functions become infinite for $x=0$, and by the greater regularity of the formulæ, which is mainly due to the absence of k- or k'-coefficients. The formulæ relating to this group have also a special prominence, as is apparent even from the formulæ given in this paper.

ON A POINT CONNECTED WITH SYMBOLIC METHODS OF INTEGRATION.

By Prof. *W. Woolsey Johnson.*

1. If ϕ denotes a rational integral function, D the operative symbol $\frac{d}{dx}$, and X a function of x, there is no ambiguity about the meaning of the symbol $\phi(D)X$; but the value of the inverse symbol $\frac{1}{\phi(D)}X$, which represents y in the equation

$$\phi(D)\,y = X,$$

is indefinite with regard to each of the terms contained in the complementary function corresponding to $\phi(D)$; that is to say, the terms which constitute the value of y in the equation

$$\phi(D)\,y = 0,$$

each of which contains an arbitrary constant factor. Thus two legitimate values of an inverse symbol of the above form may differ by any multiple of any of the terms belonging to the complementary function corresponding to its denominator.

2. When symbols whose values are thus indefinite occur in equations, we cannot of course attribute to each of them independently any one of its legitimate values; but, usually, the equation being so written as to express the value of one of them in terms of the others, we may attribute any legitimate value to each of these others; in other words, the equation may be understood to assert that, taking any legitimate values of the symbols in the second member, the result will be a legitimate value of the symbol in the first member. When this is the case, no error can arise in the application of the equation; we have only to bear in mind the indefinite character of the symbol whose value is required.

3. The formulæ commonly employed for the reduction of these inverse symbols, however, have not always this character, so that errors may arise from inattention to this peculiarity.

For example, the extended Leibnitz's theorem,

$$\phi(D)\,xV = x\phi(D)\,V + \phi'(D)\,V,$$

when applied to the inverse symbol becomes

$$\frac{1}{\phi(D)}\,xV = x\frac{1}{\phi(D)}\,V - \frac{\phi'(D)}{\{\phi(D)\}^2}\,V.$$

The terms of the complementary function by which the value of the symbol $\frac{\phi'(D)}{\{\phi(D)\}^2}V$ may be arbitrarily altered consist not only of the terms of the complementary function of $\phi(D)$, with respect to which the meaning of the symbol in the first member is indeterminate, but of as many other terms, which, being the products of these terms by x, are in fact the arbitrary terms in the values of $x\frac{1}{\phi(D)}V$; hence an arbitrary alteration in one of the latter set of terms must be accompanied by a corresponding (not an independently arbitrary) alteration in the meaning of the symbol $\frac{1}{\phi(D)}V$.

4. In practice, the values assigned to the inverse symbols in the second member will usually be their "simplest values"; but, this being done, we cannot be sure that the result is a legitimate value of the symbol in the first member; it may fail of this in respect to a multiple of one of the higher terms, so to speak, of the complementary function of $\{\phi(D)\}^2$. For example, in solving the equation

$$(D^2+1)\,y = x\sin x \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we have

$$y = \frac{1}{D^2+1}\,x\sin x = x\frac{1}{D^2+1}\sin x - \frac{2D}{(D^2+1)^2}\sin x \ldots(2),$$

and here the uncertainty of the result respects terms of the forms $x\sin x$ and $x\cos x$.

Now the simplest value of $\frac{1}{D^2+1}\sin x$ is $-\tfrac{1}{2}x\cos x$,

and " " " $\frac{1}{(D^2+1)^2}\sin x$ " $-\tfrac{1}{8}x^2\sin x$;

if these values are substituted in equation (2), we have

$$y = -\tfrac{1}{2}x^2 \cos x + \tfrac{1}{4}D\,(x^2 \sin x)$$
$$= -\tfrac{1}{4}x^2 \cos x + \tfrac{1}{2}x \sin x,$$

which is wrong. Again, if we reverse the order of the operations in the last term of equation (2), and write

$$y = x\,\frac{1}{D^2+1}\sin x - 2\,\frac{1}{(D^2+1)^2}\cos x;$$

then, the simplest value of $\dfrac{1}{(D^2+1)^2}\cos x$ being $-\tfrac{1}{8}x^2\cos x$, we have

$$y = -\tfrac{1}{2}x^2 \cos x + \tfrac{1}{4}x^2 \cos x = -\tfrac{1}{4}x^2 \cos x,$$

which is also wrong, the true value of y in equation (1) being

$$y = -\tfrac{1}{4}x^2 \cos x + \tfrac{1}{4}x \sin x.$$

5. The two values of the symbol $\dfrac{D}{(D^2+1)^2}\sin x$, employed above, are of course both legitimate values of that symbol, differing as they do by a term in the complementary function corresponding to its denominator; the difficulty results simply from the introduction of a symbol whose denominator is higher in degree than that of the symbol which it is proposed to evaluate.

6. We sometimes have occasion to raise the degree of a fractional symbol by multiplying the numerator and denominator by a common factor, as in the process

$$\frac{1}{D-1}\sin x = \frac{D+1}{D^2-1}\sin x = -\tfrac{1}{2}(D+1)\sin x = -\tfrac{1}{2}(\cos x + \sin x).$$

In this case, no uncertainty can arise if we take care to perform the direct operation last; in other words, $\dfrac{1}{D-1}X$ is not necessarily equivalent to $\dfrac{1}{D^2-1}(D+1)X$, but it is equivalent to $(D+1)\dfrac{1}{D^2-1}X$; for, although two legitimate values of $\dfrac{1}{D^2-1}X$ may differ by a multiple of e^{-x}, (which is *not* an indefinite term of the symbol $\dfrac{1}{D-1}X$,) this difference is annulled by the direct operation $D+1$.

7. To return to the extended Leibnitz' Theorem, when the two terms are reduced by the general method to expression by means of the integral sign, as for example in the process

$$\frac{1}{D-3}x\log x = x\frac{1}{D-3}\log x - \frac{1}{(D-3)^2}\log x$$

$$= xe^{3x}\int e^{-3x}\log x\,dx - e^{3x}\iint e^{-3x}\log x\,dx^2 \,\ldots\ldots\ldots(3),$$

the final result is the same as if the general method had been employed at once, and the result reduced by "integration by parts." Here the same difficulty inheres in the final expression; thus, in equation (3), the three integral signs would seem to imply three independent arbitrary constants instead of only one. To remove the ambiguity, it is however only necessary to understand that, whatever be the particular value of the integral $\int e^{-3x}\log x\,dx$ in the first term of the third member, the same value must be attributed to it in the second term; just as, in the formula

$$\int u\,dv = uv - \int v\,du,$$

v in the first term of the second member is a particular value of $\int dv$, and the same value must be used in evaluating $\int v du$.

8. So also, in equation (2), Art. 4, if, having assigned the particular value of $\frac{1}{D^2+1}\sin x$, viz., $-\frac{1}{2}x\cos x$, we had employed this value in determining the particular value of $\frac{1}{(D^2+1)^2}\sin x$ proper to be used in this equation, we should have obtained the correct result. Thus

$$\frac{1}{(D^2+1)^2}\sin x = -\frac{1}{2}\frac{1}{D^2+1}x\cos x$$

$$= -\tfrac{1}{2}\text{ real part of } e^{ix}\frac{1}{D(D+2i)}x$$

$$= \text{real part of } -\tfrac{1}{2}\frac{e^{ix}}{2i}\frac{1}{D}\left(1-\frac{D}{2i}\right)x$$

$$= \text{real part of } \frac{i\cos x - \sin x}{4}\left[\frac{x^2}{2} - \frac{x}{2i}\right]$$

$$= -\frac{x^2\sin x}{8} - \frac{x\cos x}{8};$$

here we should of course reject the last term if our object was simply to evaluate $\frac{1}{(D^2+1)^2}\sin x$; but, retaining it for our present purpose, and substituting in equation (2), we have

$$\frac{1}{D^2+1}\,x\sin x = -\tfrac{1}{2}x^2\cos x + \tfrac{1}{4}D\,(x^2\sin x + x\cos x)$$
$$= -\tfrac{1}{4}x^2\cos x + \tfrac{1}{4}x\sin x + \tfrac{1}{4}\cos x,$$

of which the last term is rejected as belonging to the complementary function.

Of course equation (1) is itself more readily solved by the process employed above in evaluating $\frac{1}{D^2+1}\,x\cos x$.

A PROOF OF FOURIER'S SERIES THEOREM.

By *A. R. Johnson.*

The dissimiliarity of the following proof to the many demonstrations in existence, and the points of interest that readily flow from it, may perhaps suffice to excuse its publication.

Throughout the language of hyper-tridimensional space is used when convenient.

Take any function $f(x_1, x_2, \ldots, x_{n-1})$ of the $n-1$ variables $x_1, x_2, \ldots, x_{n-1}$. Then

$$\text{Lt.}_{\zeta=0}\int\frac{\zeta f(x_1, x_2, \ldots, x_{n-1})\,dx_1dx_2\ldots dx_n}{\{\zeta^2+(x_1-x_1')^2+(x_2-x_2')^2+\ldots+(x_{n-1}-x'_{n-1})^2\}^{\frac{1}{2}n}}\ldots(1)$$

is $\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}f(x'_1, x'_2, \ldots, x'_{n-1})$ if the point $x'_1, x'_2, \ldots, x'_{n-1}$ be within the space of integration, and zero if without; the function f being supposed continuous throughout this space. The proof of this follows at once from the transformation formulæ employed by Green ("Attractions of Ellipsoids of Varying Density," *Collected Papers*), and by Cayley in his "Memoir on Prepotentials" (*Phil Trans.* CLXIII.).

Now take for the limits of the space of integration

$$x_1 = \pm l_1, \quad x_2 = \pm l_2, \quad \ldots, \quad x_{n-1} = \pm l_{n-1},$$

and let $x'_1, x'_2, \ldots, x'_{n-1}$ lie within it. Then, if $r_1, r_2, \ldots, r_{n-1}$ denote positive or negative integers, the point

$$x_1 - 2r_1l_1, \quad x_2 - 2r_2l_2, \quad \ldots, \quad x' - 2r_{n-1}l_{n-1}$$

lies without the space of integration, so that

$$\mathrm{Lt.}_{\zeta=0}\int_{-l_1}^{+l_1}\int_{-l_2}^{+l_2}\cdots\int_{-l_{n-1}}^{+l_{n-1}}$$
$$\frac{\zeta\,dx_1\,dx_2\ldots dx_{n-1}f(x_1,\,x_2,\,\ldots,\,x_{n-1})}{\{\zeta^2+(x_1-x'_1+2r_1l_1)^2+\ldots+(x_{n-1}-x'_{n-1}+2r_{n-1}l_{n-1})^2\}^{\frac{1}{2}n}}$$

is zero except when all the r's vanish, and therefore

$$\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}f(x'_1,\,x'_2,\,\ldots,\,x'_{n-1})$$
$$=\mathrm{Lt.}_{\zeta=0}\int_{-l_1}^{+l_1}\int_{-l_2}^{+l_2}\cdots\int_{-l_{n-1}}^{+l_{n-1}}\sum_{r_1=-\infty}^{r_1=+\infty}\sum_{r_2=-\infty}^{r_2=+\infty}\cdots\sum_{r_{n-1}=-\infty}^{r_{n-1}=+\infty}$$
$$\frac{\zeta\,dx_1\,dx_2\ldots dx_{n-1}f(x_1,\,x_2,\,\ldots,\,x_{n-1})}{\{\zeta^2+(x_1-x'_1+2r_1l_1)^2+\ldots+(x_{n-1}-x'_{n-1}+2r_{n-1}l_{n-1})^2\}^{\frac{1}{2}n}}\ldots(2).$$

Now the expression

$$\sum_{r_1=-\infty}^{r_1=+\infty}\sum_{r_2=-\infty}^{r_2=+\infty}\cdots\sum_{r_{n-1}=-\infty}^{r_{n-1}=+\infty}$$
$$\frac{\zeta}{\{\zeta^2+(v_1+2r_1l_1)^2+\ldots+(v_{n-1}+2r_{n-1}l_{n-1})^2\}^{\frac{1}{2}n}}\ldots(3)$$

is unaltered when any of the v's are increased or diminished by any even multiples of the corresponding l's or when the signs of the v's are changed. Such a function is appropriately represented by a sum of products of cosines of multiples of $\frac{\pi v_1}{l_1},\ \frac{\pi v_2}{l_2},\ \ldots,\ \frac{\pi v_{n-1}}{l_{n-1}}$.

Suppose

$$A\cos\frac{m_1\pi v_2}{l_1}\cos\frac{m_2\pi v_2}{l_2}\ldots\cos\frac{m_{n-1}\pi v_{n-1}}{l_{n-1}}$$

a term in this representation. We then have

$$\sum_{r=-\infty}^{r=+\infty}\frac{\zeta\cos\frac{m_1\pi v_1}{l_1}\cos\frac{m_2\pi v_2}{l_2}\ldots\cos\frac{m_{n-1}\pi v_{n-1}}{l_{n-1}}}{\{\zeta^2+(v_1+2r_1l_1)^2+\ldots+(v_{n-1}+2r_{n-1}l_{n-1})^2\}^{\frac{1}{2}n}}$$
$$=A\cos^2\frac{m_1\pi v_1}{l_1}\cos^2\frac{m_2\pi v_2}{l_2}\ldots\cos^2\frac{m_{n-1}\pi v_{n-1}}{l_{n-1}}$$

+ products of cosines in which every factor is not a square.

Integrating with respect to $v_1, v_2, \ldots, v_{n-1}$ between the limits $\pm l_1, \pm l_2, \ldots, \pm l_{n-1}$,

$$Al_1 l_2 \ldots l_{n-1} = \int_{-l_1}^{+l_1} \int_{-l_2}^{+l_2} \cdots \int_{l_{n-1}}^{l_{n-1}} \sum_{r=-\infty}^{r=+\infty} \frac{dv_1\, dv_2 \ldots dv_{n-1}\, \zeta \cos \frac{m_1 \pi v_1}{l_1} \ldots \cos \frac{m_{n-1} v_{n-1}}{l_{n-1}}}{\{\zeta^2 + (v_1 + 2r_1 l_1)^2 + \ldots + (v_{n-1} + 2r_{n-1} l_{n-1})^2\}^{\frac{1}{2}n}},$$

introducing however a factor 2 for every evanescent m into the left-hand member. It is not necessary here to calculate the right-hand member though this is easily done, but merely to find its limit when ζ vanishes. This, by (1), is $\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}$. Hence

$$\text{Lt.}_{\zeta=0}\, A = \frac{K}{l_1 l_2 \ldots l_{n-1}},$$

where it is necessary to introduce a divisor 2 on the right-hand for every evanescent m in the term multiplied by A, and $K = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}$.

It is obvious that in the representation of (3), every possible variation of the m's must occur. For, if one were wanting, on multiplying by it and integrating as before, we should have

$$0 = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}.$$

The expression (3) is equal then to

$$\sum_{m_1=-\infty}^{m_1=+\infty} \cdots \sum_{m_{n-1}=-\infty}^{m_{n-1}=+\infty} \frac{KB}{l_1 l_2 \ldots l_{n-1}} \cos \frac{m_1 \pi v_1}{l_1} \ldots \cos \frac{m_{n-1} \pi v_{n-1}}{l_{n-1}},$$

when B is a function of ζ which becomes 2^{-n+1} when ζ vanishes. (It may be shewn that

$$\sum_{r=-\infty}^{r=+\infty} \{\zeta^2 + (v_1 + 2r_1 l_1)^2 + \ldots + (v_{n-1} + 2r_{n-1} l_{n-1})^2\}^{\frac{1}{2}n}$$

$$= \frac{\pi^{\frac{1}{2}n} 2^{-n+1}}{\Gamma(\frac{1}{2}n)\, \zeta l_1 l_2 \ldots l_{n-1}} \sum_{m=-\infty}^{m=+\infty} e^{-\pi\zeta\kappa} \cos \frac{m_1 \pi x_1}{l_1} \ldots \cos \frac{m_{n-1} \pi x_{n-1}}{l_{n-1}},$$

where $$\kappa^2 = \frac{m_1^2}{l_1^2} + \frac{m_2^2}{l_2^2} + \ldots + \frac{m^2_{n-1}}{l^2_{n-1}},$$

a rapidly convergent series).

Substituting then in (2) we have

$$f(x'_1, x'_2, \ldots, x'_{n-1}) = \frac{2^{-n+1}}{l_1 l_2 \ldots l_{n-1}} \sum_{m=-\infty}^{m=+\infty} \int_{-l_1}^{+l_1} \int_{-l_2}^{+l_2} \ldots \int_{-l_{n-1}}^{+l_{n-1}}$$

$$f(x_1, x_2, \ldots, x_{n-1}) \cos \frac{m_1 \pi}{l_1}(x_1 - x'_1) \ldots \cos \frac{m_{n-1}\pi}{l_{n-1}}(x_{n-1} - x'_{n-1}) dx_1 \ldots dx_{n-1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

2. It has hitherto been assumed that $f(x_1, x_2, \ldots, x_{n-1})$ is continuous at the point $x'_1, x'_2, \ldots, x'_{n-1}$. Now suppose this point to be the meeting place of s regions, in going from one of which to another $f(x_1, x_2, \ldots, x_{n-1})$ becomes discontinuous.

It is still true that the Fourier Series is equal to the integral

$$\pi^{-\frac{1}{2}n} \Gamma(\tfrac{1}{2}n) \,\mathrm{Lt.}_{\zeta=0} \int_{-l_1}^{+l_1} dx_1 \int_{-l_2}^{+l_2} dx_2 \ldots \int_{-l_{n-1}}^{+l_{n-1}} dx_{n-1} \sum_{r=-\infty}^{r=+\infty}$$

$$\frac{\zeta f(x_1, x_2, \ldots, x_{n-1})}{\{\zeta^2 + (x_1 - x'_1 + 2r_1 l_1)^2 + \ldots + (x'_{n-1} - x'_{n-1} + 2r_{n-1} l_{n-1})^2\}^{\frac{1}{2}n}},$$

or if the point $x'_1, x'_2, \ldots, x'_{n-1}$ be within the space of integration to

$$\pi^{-\frac{1}{2}n} \Gamma(\tfrac{1}{2}n) \,\mathrm{Lt.}_{\zeta=0} \int_{-l_1}^{+l_1} dx_1 \ldots \int_{-l_{n-1}}^{+l_{n-1}} dx_{n-1}$$

$$\frac{\zeta f(x_1, x_2, \ldots, x_{n-1})}{\{\zeta^2 + (x_1 - x'_1 + 2r_1 l_1)^2 + \ldots + (x_{n-1} - x'_{n-1} + 2r_{n-1} l_{n-1})^2\}^{\frac{1}{2}n}}.$$

But since the only sensible parts of this integral are those contributed by the elements for which $x_1, x_2, \ldots, x_{n-1}$ is immediately near $x'_1, x'_2, \ldots, x'_{n-1}$, $f(x_1, x_2, \ldots, x_{n-1})$ may be supposed to have the value that it has at a point in the immediate vicinity of $x'_1, x'_2, \ldots, x'_{n-1}$ in any region, in the elements contributed by points in that region. The integral is therefore equal to

$$\pi^{-\frac{1}{2}n} \Gamma(\tfrac{1}{2}n) f_1(x'_1, x'_2, \ldots, x'_{n-1}) \,\mathrm{Lt.}_{\zeta=0}$$

$$\int \frac{\zeta \, dx_1 \ldots dx_{n-1}}{\{\zeta^2 + (x_1 - x'_1)^2 + \ldots + (x_{n-1} - x'_{n-1})^2\}^{\frac{1}{2}n}}$$

$$+ \pi^{-\frac{1}{2}n} \Gamma(\tfrac{1}{2}n) f_2(x'_1, x'_2, \ldots, x'_{n-1}) \,\mathrm{Lt.}_{\zeta=0}$$

$$\int \frac{\zeta \, dx_1 \ldots dx_{n-1}}{\{\zeta^2 + (x_1 - x'_1)^2 + \ldots + (x_{n-1} - x'_{n-1})^2\}^{\frac{1}{2}n}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \pi^{-\frac{1}{2}n} \Gamma(\tfrac{1}{2}n) f_s(x'_1, x'_2, \ldots, x'_{n-1}) \,\mathrm{Lt.}_{\zeta=0}$$

$$\int \frac{\zeta \, dx_1 \ldots dx_{n-1}}{\{\zeta^2 + (x_1 - x'_1)^2 + \ldots + (x_{n-1} - x'_{n-1})^2\}^{\frac{1}{2}n}},$$

f_r being the value of f in the r^{th} region, and wherein the coefficient of $f_r(x'_1, x'_2, \ldots, x'_{n-1})$ the integration is to extend in the first place through the space in the r^{th} region immediately near $x'_1, x'_2, \ldots, x'_{n-1}$; but taking into account the fact that the only elements of the integral which are sensible are those for points indefinitely near $x'_1, x'_2, \ldots, x'_{n-1}$, we may effect the integration over all space, extending to infinity from $x'_1, x'_2, \ldots, x'_{n-1}$ in all directions lying primarily in the region.

The value of the series is then

$$\Omega_1 f_1(x'_1, x'_2, \ldots, x'_{n-1}) + \Omega_2 f_2(x'_1, x'_2, \ldots, x'_{n-1}) + \ldots \\ + \Omega_s f_s(x'_1, x'_2, \ldots, x'_{n-1}) \ldots (B),$$

where $\Sigma\Omega = 1$, and

$$\Omega_r = \pi^{-\frac{1}{2}n}\,\Gamma\left(\tfrac{1}{2}n\right)\int \frac{dx_1\,dx_2\ldots dx_{n-1}}{\{1 + x_1^2 + x_2^2 + \ldots + x_{n-1}^2\}^{\frac{1}{2}n}},$$

taken as just described.

In space of three dimensions Ω_r is the ratio to 4π of the solid angle of the cone which at $x'_1, x'_2, \ldots, x'_{n-1}$ touches the r^{th} region, and in space of two dimensions the ratio to 2π of the angle between the tangents at $x'_1, x'_2, \ldots, x'_{n-1}$ to the r^{th} region.

3. We are now in a position to ascertain the value of the series (A) when some of the x's are equal to the corresponding l's, noticing, in (B), that $f_r(x'_1, \ldots, x'_{n-1})$ is the value of the Fourier Series at a point immediately near $x'_1, x'_2, \ldots, x'_{n-1}$ but entirely within the r^{th} region. The series for a point $x'_1 x'_2 \ldots x'_{n-1}$ is equal to that for the point

$$x'_1 \pm 2r_1 l_1,\ x'_2 \pm 2r_2 l_2,\ \ldots,\ x'_{n-1} \pm 2r'_{n-1} l_{n-1}.$$

The value of the series for the point

$$l_1 + h_1,\ l_2 + h_2,\ \ldots,\ x'_r,\ \ldots,\ x'_{n-1},$$

is therefore equal to that for the point

$$-(l_1 - h_1),\ -(l_2 - h_2),\ \ldots,\ x'_r,\ \ldots,\ x'_{n-1},$$

h_1, h_2, &c. being small positive quantities. But this last point is within the space of integration, and the value of the series is therefore

$$f\{-(l_1 - h_1),\ -(l_2 - h_2),\ \ldots,\ x'_r,\ \ldots,\ x'_{n-1}\},$$

which approaches the value $f(-l_1, -l_2, \ldots, x'_r, \ldots, x'_{n-1})$ when $h_1, h_2, \ldots, h_{n-1}$ are made indefinitely small.

4. It was proved in § 1 that at any point $x'_1 x'_2 \dots x'_{n-1}$ the value of the series is

$$\pi^{-\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\,\text{Lt.}_{\zeta=0}\sum_{r=-\infty}^{r=+\infty}\int_{-l_1}^{+l_1}\int_{-l_2}^{+l_2}\dots\int_{-l_{n-1}}^{+l_{n-1}}\frac{\zeta f(x_1, x_2, \dots, x_{n-1})\,dx_1 dx_2 \dots dx_{n-1}}{[\zeta^2 + (x_1 - x'_1)^2 + \dots + (x_{n-1} - x'_{n-1} + 2r_{n-1}l_{-1})^2]^{\frac{1}{2}n}},$$

and this is

$$\pi^{-\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\,\text{Lt.}_{\zeta=0}\sum_{r=-\infty}^{r=+\infty}\int_{(2r_1-1)l_1}^{(2r_1+1)l_1}\int_{(2r_2-1)l_2}^{(2r_2+1)l_2}\dots\int_{(2r_{n-1}-1)l_{n-1}}^{(2r_{n-1}+1)l_{n-1}}$$

$$dx_1 dx_2 \dots dx_{n-1}\frac{\zeta f(x_1 - 2rl_1, x_2 - 2r_2 l_2, \dots)}{[\zeta^2 + (x_1 - x'_1)^2 + \dots + (x_{n-1} - x'_{n-1})^2]^{\frac{1}{2}n}},$$

or $\pi^{-\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)\,\text{Lt.}_{\zeta=0}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}$

$$\frac{\zeta F(x_1, x_2, \dots, x_{n-1})\,dx_1 dx_2 \dots dx_{n-1}}{[\zeta^2 + (x_1 - x_1')^2 + \dots + (x_{n-1} - x'_{n-1})^2]^{\frac{1}{2}n}}.$$

where F is a discontinuous function, such that, if $x_1, x_2, \dots, x_{n-1}$ lie within the limits

$$(2r_1 \pm l_1),\ (2r_2 \pm 1)\, l_2,\ \dots,\ (2r_{n-1} \pm 1)\, l_{n-1},$$

$$F\{x_1, x_2, \dots, x_{n-1}\} = f\{x_1 - 2r_1 l_1, \dots, x_{n-1} - 2r_{n-1}l_{n-1}\}.$$

This result justifies the application of the results arrived at above in the case of discontinuity at a point within the space defined by the limits $x_1 = \pm l_1$, $x_2 = \pm l_2, \dots$ to the case of discontinuity at a point on the boundary of this region or elsewhere.

5. In continuation of § 3, suppose that

$$x'_1 = l_1,\ x'_2 = l_2,\ \dots,\ x'_s = l_s,$$

then, at the point $x', x'_2, \dots, x'_{n-1}$, space will be divided into 2^s regions, in which, at points near $x'_1, x'_2, \dots, x'_{n-1}$, the series has different values, viz. one region in which $x_1 > l_1$, $x_2 < l_2$, $x_3 < l_3$, &c., and in which the series has the value $f(-l_1, l_2, \dots)$, and so on.

Also, obviously,

$$\Omega_1 = \Omega_2 = \dots = \Omega 2^s = 2^{-s},$$

and therefore the value of the series at the point

$$l_1, l_2, \dots, l_s, x'_{s+1}, \dots, x'_{r-1}$$

is $$2^{-s}\Sigma f(\pm l_1, \pm l_2, \dots, \pm l_s, x'_{s+1}, \dots, x'_{n-1}),$$

where Σ denotes that the sum is to be taken of all terms obtained by taking all possible variations of the signs ±.

6. To illustrate the above discussion of the value of the Fourier Series in the case of discontinuity, suppose $f(x, y, z)$ equal to $f_1(x, y, z)$ within a cylinder whose axis is that of z, and bounding planes $z=0$, $z=c$, and let in all space external to the cylinder for which z is positive, $f(x, y, z)=f_2(x, y, z)$, and when z is negative $f(x, y, z)=f_3(x, y, z)$, the whole space being comprised within the limits

$$x=\pm l_1, \quad y=\pm l_2, \quad z=\pm l_3.$$

Then:

(1) At any point in the part of the plane $z=c$, intercepted by the cylinder, the series $=\frac{1}{2}f_1+\frac{1}{2}f_2=s_1$ say.

(2) At any point of the upper rim, series $=\frac{1}{4}f_1+\frac{3}{4}f_2=s_2$.

(3) At any point in the convex surface, series $=\frac{1}{2}f_1+\frac{1}{2}f_2=s_3$.

(4) At any point in the lower rim, series $=\frac{1}{4}f_1+\frac{1}{4}f_2+\frac{1}{2}f_3=s_4$.

(5) At any point in the part of plane $z=0$, intercepted by the cylinder, series $=\frac{1}{2}f_1+\frac{1}{2}f_3=s_5$.

(6) At any other point in the plane of z the value is $\frac{1}{2}f_2+\frac{1}{2}f_3=s_6$.

7. If we attach numerical values to f_1, f_2, f_3 in § 6, such that $s_1, s_2, s_3, s_4, s_5, s_6$ are all different, the series if equated to s_2 gives the equation to the upper rims periodically recurring; if to s_1 there results the equation to a series of circular discs arranged symmetrically; if to s_6 to a series of planes arranged parallelwise and having circular holes punched in them at regular intervals; and so on.

In space of three dimensions values for the series are easily obtained in the case of the sphere and parallelipiped, putting $f=1$ inside and $=0$ outside. The general terms of the series are: for the sphere of radius a

$$\frac{64}{\pi^2 l_1 l_2 l_3}\left[\frac{1}{\kappa^3}\sin\pi a\kappa-\frac{\pi a}{\kappa^2}\cos\pi a\kappa\right]\cos\frac{m_1\pi x}{l_1}\cos\frac{m_2\pi y}{l_2}\cos\frac{m_3\pi z}{l_3},$$

and for the parallelipiped of dimensions $2a_1$, $2a_2$, $2a_3$,

$$\frac{8}{\pi^3}\frac{1}{m_1 m_2 m_3}\sin\frac{m_1\pi a_1}{l_1}\sin\frac{m_2\pi a_2}{l_2}\sin\frac{m_3\pi a_3}{l_3}$$

$$\times\cos\frac{m_1\pi x}{l_1}\cos\frac{m_2\pi y}{l_2}\cos\frac{m_3\pi z}{l_3}.$$

Equating these to 1, we get the equation to the volumes; to $\frac{1}{2}$, to the surfaces of the spheres and to the faces of the parallelipipeds. Equating the second series to $\frac{1}{4}$ and $\frac{1}{8}$ to the edges and vertices of the parallelipiped respectively. The integrations for the circle and rectangle and for the finite straight line are also easily effected.

8. In conclusion it may be remarked that Art. 4 establishes the relation

$$\sum_{r=-\infty}^{r=+\infty} F(x_1 + 2r_1 l_1, x_2 + 2r_2 l_2, \ldots, x_{n-1} + 2r_{n-1} l_{n-1})$$

$$= \frac{1}{l_1 l_2 l_3 \ldots l_{n-1}} \sum_{m=-\infty}^{m=+\infty} \int_{-\infty}^{+\infty} dx'_1 \int_{-\infty}^{+\infty} dx'_2 \ldots \int_{-\infty}^{+\infty} dx'_{n-1}$$

$$F(x'_1, x'_2, \ldots, x'_{n-1}) \cos \frac{m_1 \pi (x_1 - x'_1)}{l_1} \cos \frac{m_2 \pi (x_2 - x'_2)}{l_2}$$

$$\ldots \cos \frac{m_{n-1} \pi (x_{n-1} - x'_{n-1})}{l_{n-1}},$$

whence may be deduced, by putting

$$F(x_1, x_2, \ldots, x_{n-1}) = e\{- x_1^2 - x_2^2, \ldots, - x^2_{n-1}\},$$

the Fourier series equivalent to a particular case of the general Göpel function, viz.

$$\sum_{r=-\infty}^{r=+\infty} \left[e - \sum_{i=1}^{i=n-1} (x_i + 2r_i l_i)^2 \right]$$

$$= \frac{\sqrt{\pi}}{2 l_1 l_2 \ldots l_{n-1}} \sum_{m=-\infty}^{m=+\infty} e^{-\kappa^2} \cos \frac{m_1 \pi x'_1}{l_1} \cos \frac{m_2 \pi x'_2}{l_2} \ldots \cos \frac{m_{n-1} \pi x'_{n-1}}{l_{n-1}}$$

where $$\frac{\kappa^3}{\pi^2} = \frac{m_1^2}{l_1^2} + \frac{m_2^2}{l_2^2} + \ldots + \frac{m^2_{n-1}}{l^2_{n-1}},$$

and generally

$$\sum_{r=-\infty}^{r=+\infty} f\{ \sum_{i=1}^{i=n-1} (x_i + 2r_i l_i)^2 \}$$

$$= \frac{2\pi^{\frac{1}{2}(n-2)}}{\Gamma \frac{1}{2}(n-2)} \frac{1}{l_1 l_2 \ldots l_{n-1}} \sum_{m=-\infty}^{m=+\infty} \cos \frac{m_1 \pi x_1}{z_1} \ldots \cos \frac{m_{n-1} \pi x_{n-1}}{z_{n-1}}$$

$$\int_{-\infty}^{+\infty} \int_0^{+\infty} dx\, dr \left[(\zeta^2 + x^2)^{\frac{1}{2}(n-2)} r^{n-3} f\{(\zeta^2 + x^2)(1 + r^2)\} \cos \kappa x. \right.$$

Whence

$$\sum_{r=-\infty}^{r=+\infty} \{\zeta^2 + (x_1 + 2r_1 l_1)^2 + \ldots + (x_{n-1} + 2r_{n-1} l_{n-1})^2\}^{-\frac{1}{2}n-s}$$

$$= \frac{\pi^{\frac{1}{2}(n-2)}\,\Gamma\left(\tfrac{1}{2}n\right)\Gamma(s)}{\Gamma\,\tfrac{1}{2}(n-2)\,\Gamma\left(\tfrac{1}{2}n+s\right)} \sum_{m=-\infty}^{m=+\infty} \frac{1}{l_1 l_2 \ldots l_{n-1}} \cos\frac{m_1 \pi x_1}{l_{n-1}}$$

$$\ldots \cos\frac{m_{n-1}\pi x_{n-1}}{l_{n-1}}\,\zeta^{-s-1} \int_{-\infty}^{+\infty} \frac{\cos\kappa\zeta x}{(1+x^2)^s}\,dx.$$

which depends on the integral $\int_{-\infty}^{+\infty} \frac{\cos(\kappa\zeta x)}{(1+x^2)^s}\,dx$, that is, the product of a Bessel's Function of the second kind with argument $\kappa\zeta$ and of order $s+\frac{1}{2}$ by $(\kappa\zeta)^{s+\frac{1}{2}}$.

P.S.—In Mr. Alexander's paper in the August number of the *Messenger*, the objection to the series on p. 61 is that the series is divergent.* In functional series, other than trigonometric, the defective series is not necessarily inappropriate for the representation of a function.

ORTHOGONAL SYSTEMS OF CIRCLES.

By *R. Lachlan, B.A.*, Fellow of Trinity College, Cambridge.

BY an orthogonal system of circles is usually meant a system of four circles, which are such that each cuts the other three orthogonally. The chief properties of such a system are; (1) their centres are such that anyone is the orthocentre of the other three, (2) the sum of the squares of the reciprocals of their radii is zero, and (3), if the circles be denoted by (1, 2, 3, 4) the power of any other circles (x, y) is given by

$$\pi_{x,y} = \frac{\pi_{x,1}\cdot\pi_{y,1}}{\pi_{1,1}} + \frac{\pi_{x,2}\cdot\pi_{y,2}}{\pi_{2,2}} + \frac{\pi_{x,3}\cdot\pi_{y,3}}{\pi_{3,3}} + \frac{\pi_{x,4}\cdot\pi_{x,4}}{\pi_{4,4}}.$$

(2) being really a particular case of (3).

In this paper I propose, to define two systems, each containing four circles, as orthogonal systems, when either system being given, the second system is derived by taking the circles orthogonally to the given ones three at a time

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* See *Messenger*, vol. XV. pp. 81–87.

It will be seen that two such systems have properties as simple as those mentioned above for the particular case when the two systems are identical. The chief properties are are mentioned in a paper, communicated to the Royal Society; an abstract of which will be found in the Proceedings for 1886 p. 242.; but I propose to reproduce them here, for the sake of some examples which are of some interest.

Preliminary, §§ 1–5.

1. If the distance between two circles whose radii are r, r' be d; the expression $d^2 - r^2 - r'^2$ is called the *power* of the two circles. Denoting this by π, we have $\pi = 2rr' \cos\omega$ where ω is the angle of intersection of the two circles.

2. The powers of any ten circles are connected by the relation—

$$\begin{vmatrix} \pi_{1,6}, & \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10} \\ \pi_{2,6}, & \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10} \\ \pi_{3,6}, & \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10} \\ \pi_{4,6}, & \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10} \\ \pi_{5,6}, & \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10} \end{vmatrix} = 0,$$

which is conveniently expressed thus—

$$\pi \begin{pmatrix} 1, & 2, & 3, & 4, & 5 \\ 6, & 7, & 8, & 9, & 10 \end{pmatrix} = 0.$$

To prove this: let the equations of any two circles be

$$x^2 + y^2 + 2\alpha x + 2\beta y + c = 0$$
$$x^2 + y^2 + 2\alpha' x + 2\beta' y + c' = 0,$$

then their power is given by

$$\pi = c + c' - 2\alpha\alpha' - 2\beta\beta'.$$

And the above relation follows at once by multiplying together the matrices—

$$\begin{vmatrix} c_1, & -2\alpha_1, & -2\beta_1, & 1 \\ c_2, & -2\alpha_2, & -2\beta_2, & 1 \\ c_3, & -2\alpha_3, & -2\beta_3, & 1 \\ c_4, & -2\alpha_4, & -2\beta_4, & 1 \\ c_5, & -2\alpha_5, & -2\beta_5, & 1 \end{vmatrix} \quad \begin{vmatrix} 1, & \alpha_6, & \beta_6, & c_6 \\ 1, & \alpha_7, & \beta_7, & c_7 \\ 1, & \alpha_8, & \beta_8, & c_8 \\ 1, & \alpha_9, & \beta_9, & c_9 \\ 1, & \alpha_{10}, & \beta_{10}, & c_{10} \end{vmatrix}.$$

3. The general theorem includes the case where any of the circles are replaced by straight lines, or the line at infinity: provided that we define the power of a straight line and a circle to be the perpendicular from its centre on the straight line, and the power of two straight lines to be the cosine of the angle between them. Also if θ denote the line at infinity, S any circle, and L any straight line, we must have—

$$\pi_{\theta,S}=1,\ \pi_{\theta,L}=0,\ \pi_{\theta,\theta}=0.$$

4. Proceeding as in § 3, we have—

$$\pi\begin{pmatrix}\theta, 1, 2, 3\\ \theta, 4, 5, 6\end{pmatrix}=\begin{vmatrix}1, & 0, & 0, & 0\\ c_1, & -2\alpha_1, & -2\beta_1, & 1\\ c_2, & -2\alpha_2, & -2\beta_2, & 1\\ c_3, & -2\alpha_3, & -2\beta_3, & 1\end{vmatrix}\times\begin{vmatrix}0, & 0, & 0, & 1\\ 1, & \alpha_4, & \beta_4, & c_4\\ 1, & \alpha_5, & \beta_5, & c_5\\ 1, & \alpha_6, & \beta_6, & c_6\end{vmatrix}$$

$$=-16\Delta(1, 2, 3)\,.\,\Delta(4, 5, 6).$$

where $\Delta(1, 2, 3)$ denotes the area of the triangle formed by the centres of the circles (1, 2, 3).

5. If the circle (y) be the inverse of any circle (x) with respect to the circle S, it is easily proved geometrically that

$$\frac{2}{\pi_{s,s}}=\frac{1}{\pi_{x,s}}+\frac{1}{\pi_{y,s}}.$$

Formulæ for an Orthogonal System, §§ 6–10.

6. The system of circles (5, 6, 7, 8) being orthogonal to the system (1, 2, 3, 4), the theorem in § 3,

$$\pi\begin{pmatrix}x, 1, 2, 3, 4\\ y, 5, 6, 7, 8\end{pmatrix}=0$$

becomes

$$\pi_{x,y}=\frac{\pi_{x,5}\cdot\pi_{y,1}}{\pi_{1,5}}+\frac{\pi_{x,6}\cdot\pi_{y,2}}{\pi_{2,6}}+\frac{\pi_{x,7}\cdot\pi_{y,3}}{\pi_{3,7}}+\frac{\pi_{x,8}\cdot\pi_{y,4}}{\pi_{4,8}}\ \ldots(1).$$

Whence we get the following particular cases:—

Taking the line at infinity instead of the circle x,

$$1=\frac{\pi_{y,1}}{\pi_{1,5}}+\frac{\pi_{y,2}}{\pi_{2,6}}+\frac{\pi_{y,3}}{\pi_{3,7}}+\frac{\pi_{y,4}}{\pi_{4,8}}\ \ldots\ldots\ldots\ldots\ (2).$$

Taking any straight line L instead of the circle y,

$$0=\frac{\pi_{L,1}}{\pi_{1,5}}+\frac{\pi_{L,2}}{\pi_{2,6}}+\frac{\pi_{L,3}}{\pi_{3,7}}+\frac{\pi_{L,4}}{\pi_{4,8}}\ldots\ldots\ldots\ldots\ (3).$$

If this straight line be the line at infinity, this equation gives

$$0 = \frac{1}{\pi_{1,5}} + \frac{1}{\pi_{2,6}} + \frac{1}{\pi_{3,7}} + \frac{1}{\pi_{4,8}} \quad \ldots\ldots\ldots\ldots (4).$$

Also taking x and y as coincident, equation (1) gives

$$-2r_x^2 = \pi_{x,x} = \frac{\pi_{x,5}\cdot\pi_{x,1}}{\pi_{1,5}} + \frac{\pi_{x,6}\cdot\pi_{x,2}}{\pi_{2,6}} + \frac{\pi_{x,7}\cdot\pi_{x,3}}{\pi_{3,7}} + \frac{\pi_{x,8}\cdot\pi_{x,4}}{\pi_{4,8}} \quad \ldots\ldots\ldots\ldots (5),$$

or if x be a straight line,

$$-1 = \pi_{L,L} = \frac{\pi_{L,1}\cdot\pi_{L,5}}{\pi_{1,5}} + \frac{\pi_{L,2}\cdot\pi_{L,6}}{\pi_{2,6}} + \frac{\pi_{L,3}\cdot\pi_{L,7}}{\pi_{3,7}} + \frac{\pi_{L,4}\cdot\pi_{L,8}}{\pi_{4,8}} \quad \ldots\ldots\ldots\ldots (6).$$

Again from (2) we get

$$1 = \frac{\pi_{1,1}}{\pi_{1,5}} + \frac{\pi_{1,2}}{\pi_{2,6}} + \frac{\pi_{1,3}}{\pi_{3,7}} + \frac{\pi_{1,4}}{\pi_{4,8}} \quad \ldots\ldots\ldots\ldots (7).$$

7. Again from § 5, we have

$$\begin{aligned} -16\,.\,\Delta(1, 2, 3)\,.\,\Delta(5, 6, 7) &= \pi\begin{pmatrix}\theta, & 1, & 2, & 3\\ \theta, & 5, & 6, & 7\end{pmatrix} \\ &= -\pi_{1,5}\cdot\pi_{2,6}\cdot\pi_{3,7}\cdot\left(\frac{1}{\pi_{1,5}} + \frac{1}{\pi_{2,6}} + \frac{1}{\pi_{3,7}}\right) \\ &= \frac{\pi_{1,5}\cdot\pi_{2,6}\cdot\pi_{3,7}}{\pi_{4,8}} \quad \ldots\ldots\ldots\ldots (8). \end{aligned}$$

Also

$$-16\,.\,\Delta(1, 2, 3)\,.\,\Delta(6, 7, 8) = \pi\begin{pmatrix}\theta, & 1, & 2, & 3\\ \theta, & 6, & 7, & 8\end{pmatrix} = -\pi_{2,6}\cdot\pi_{3,7} \quad \ldots\ldots\ldots\ldots (9).$$

Hence

$$\frac{\Delta(6, 7, 8)}{\Delta(5, 6, 7)} = -\frac{\pi_{4,8}}{\pi_{1,5}} = \frac{\Delta(2, 3, 4)}{\Delta(1, 2, 3)} \quad \ldots\ldots\ldots (10).$$

Thus we have the theorems:—

(i) Given any four circles, the powers of each circle with respect to the corresponding circle of the orthogonal system are inversely proportional to the areas of the triangles formed by the centres of the remaining three circles.

(ii) The areas of the triangles formed their centres of any four circles are proportional to the areas formed by the centres of the corresponding circles of the orthogonal system.

9. If O_1, O_2 &c. be the centres of the circles, it is evident that if O_4 be the orthocentre of the triangle $O_1O_2O_3$, then the triangle $O_5O_6O_7$ is similar to the triangle $O_1O_2O_3$. Hence O_8 is the orthocentre of the triangle $O_5O_6O_7$.

Again if O_4 be the centroid of the triangle $O_1O_2O_3$; O_8 must be the centroid of $O_5O_6O_7$. In this case, we have—

$$\pi_{1,5}=\pi_{2,6}=\pi_{3,7}=-3\pi_{4,8} \text{ ....} (11).$$

10. Let us suppose now that the circles (2′, 3′, 4′, 5′) are the inverse circles of (2, 3, 4, 5) with respect to the circle (1); then since (6, 7, 8) are orthogonal to (1) and so invert into themselves, the system (5′, 6, 7, 8) will be the orthogonal system of (1, 2′, 3′, 4′). We have then, by (4) and (7),

$$0=\frac{1}{\pi_{1,5'}}+\frac{1}{\pi_{2',6}}+\frac{1}{\pi_{3',7}}+\frac{1}{\pi_{4',8}},$$

$$1=\frac{\pi_{1,1}}{\pi_{1,5'}}+\frac{\pi_{1,2'}}{\pi_{2',6}}+\frac{\pi_{1,3'}}{\pi_{3',7}}+\frac{\pi_{1,4'}}{\pi_{4',8}}.$$

But, by § 5,

$$\frac{2}{\pi_{1,1}}=\frac{1}{\pi_{1,5}}+\frac{1}{\pi_{1',5}}.$$

And combining these with (4) and (7) we get

$$\frac{1}{\pi_{2,6}}+\frac{1}{\pi_{2',6}}+\frac{1}{\pi_{3,7}}+\frac{1}{\pi_{3',7}}+\frac{1}{\pi_{4,8}}+\frac{1}{\pi_{4',8}}=-\frac{2}{\pi_{1,1}}=\frac{1}{r_1^2}\dots(12),$$

and

$$\frac{\pi_{1,2}}{\pi_{2,6}}+\frac{\pi_{1,2'}}{\pi_{2',6}}+\frac{\pi_{1,3}}{\pi_{3,7}}+\frac{\pi_{1,3'}}{\pi_{3',7}}+\frac{\pi_{1,4}}{\pi_{4,8}}+\frac{\pi_{1,4'}}{\pi_{4',8}}=0\dots(13).$$

Application to a rectilinear triangle, § 11.

11. Let *ABC* be a triangle, *P* any point in its plane, then regarding these four points as evanescent circles, the orthogonal system of circles consists of the circles circumscribing the four triangles formed by them.

Let O, O_1, O_2, O_3 be the centres of the four circles, R, R_1, R_2, R_3 their radii; also let x, y, z denote the distances AP, BP, CP; then we have, from (4),

$$\frac{1}{R^2-OP^2}=\frac{1}{O_1A^2-R_1^2}+\frac{1}{O_2B^2-R_2^2}+\frac{1}{O_3C^2-R_3^2}\dots(14).$$

And, from (7),

$$\left.\begin{aligned}
\frac{x^2}{R^2-OP^2} &= -1+\frac{c^2}{O_2B^2-R_2^2}+\frac{b^2}{O_3C^2-R_3^2}\\
\frac{y^2}{R^2-OP^2} &= \frac{c^2}{O_1A^2-R_1^2}-1+\frac{c^2}{O_3C^2-R_3^2}\\
\frac{z^2}{R^2-OP^2} &= \frac{b^2}{O_1A^2-R_1^2}+\frac{a^2}{O_2B^2-R_2^2}-1\\
1 &= \frac{x^2}{O_1A^2-R_1^2}+\frac{y^2}{O_2B^2-R_2^2}+\frac{z^2}{O_3C^2-R_3^2}
\end{aligned}\right\}\ldots(15).$$

But since, by (10),

$$\frac{\text{area }(ABC)}{\text{area }(PBC)}=\frac{O_1A^2-R_1^2}{R^2-OP^2}$$

the last formula may be written

$$\begin{aligned}(R^2-OP^2)\,\Delta\,(ABC) &= AP^2\,.\,\Delta\,(PBC)+BP^2\,.\,\Delta\,(PCA)\\
&\quad +CP^2\,.\,\Delta\,(PAB)\ldots\ldots\ldots\ldots\ldots\ldots(16).\end{aligned}$$

Application to a circular triangle, §§ 12–21.

12. Let ABC be a circular triangle formed by three given circles, whose centres are F, G, H. Let O be the centre of the circle which circumscribes the triangle ABC, and let E be the centre of the circle which cuts the three circles orthogonally, which may be called the *orthogonal* circle. Then clearly the points A, B, C and this orthogonal circle form a system which is orthogonal to the system which consists of the three given circles and the circum-circle of the triangle ABC.

In what follows a, b, c will be used to denote the sides of the rectilinear triangle ABC; α, β, γ to denote the angles of the circular triangle, ω the angle of intersection of the circum-circle and the orthogonal circle. The radii of the given circles will be denoted by r_1, r_2, r_3; and the radii of the circum-circle and orthogonal circle by R and r; the power of these two circles will be denoted by π. The powers of A, B, C with respect to the three given circles will be denoted by π_1, π_2, π_3; and with respect to the orthogonal circle by t_1^2, t_2^2, t_3^2.

Also if $2s = \alpha + \beta + \gamma$, we shall find it convenient to use S to denote the expression

$$\cos s . \cos(s-\alpha) . \cos(s-\beta) . \cos(s-\gamma)$$

so that $4S = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1.$

The properties of a circular triangle have been discussed in a paper which appeared in the *Quarterly Journal*, vol. XXI. pp. (1–59); in which it is shown that

$$r = \frac{r_1 r_2 r_3}{\Delta(FGH)} . S^{\frac{1}{2}} \text{ (see § 14)},$$

and that the circum-circle cuts the three given circles at angles

$$\tfrac{1}{2}\pi - (s-\alpha)\,;\ \tfrac{1}{2}\pi - (s-\beta)\,;\ \tfrac{1}{2}\pi - (s-\gamma) . \text{(see § 37)}.$$

Also $$\cos\omega = - S^{\frac{1}{2}} \sec s \text{ [see § 37, (4)]}.$$

13. From (4) we have

$$\frac{1}{\pi} + \frac{1}{\pi_1} + \frac{1}{\pi_2} + \frac{1}{\pi_3} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

From (7) we get the two sets of relations

$$\left.\begin{aligned}
1 &= \qquad\quad \frac{c^2}{\pi_2} + \frac{b^2}{\pi_3} + \frac{t_1^2}{\pi} \\
1 &= \frac{c^2}{\pi_1} \qquad\quad + \frac{a^2}{\pi_3} + \frac{t_2^2}{\pi} \\
1 &= \frac{b^2}{\pi_1} + \frac{a^2}{\pi_2} \qquad\quad + \frac{t_3^2}{\pi} \\
1 &= \frac{t_1^2}{\pi_1} + \frac{t_2^2}{\pi_2} + \frac{t_3^2}{\pi_3} - \frac{2r^2}{\pi}
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(18),$$

and

$$\left.\begin{aligned}
\frac{1}{2r_1} &= -\frac{r_1}{\pi_1} + \frac{r_2\cos\gamma}{\pi_2} + \frac{r_3\cos\beta}{\pi_3} + \frac{R\sin(s-\alpha)}{\pi} \\
\frac{1}{2r_2} &= \frac{r_1\cos\gamma}{\pi_1} - \frac{r_2}{\pi_2} + \frac{r_3\cos\alpha}{\pi_3} + \frac{R\sin(s-\beta)}{\pi} \\
\frac{1}{2r_3} &= \frac{r_1\cos\beta}{\pi_1} + \frac{r_2\cos\alpha}{\pi_2} - \frac{r_3}{\pi_3} + \frac{R\sin(s-\gamma)}{\pi} \\
\frac{1}{2R} &= \frac{r_1\sin(s-\alpha)}{\pi_1} + \frac{r_2\sin(s-\beta)}{\pi_2} \\
&\qquad + \frac{r_3\sin(s-\gamma)}{\pi_3} - \frac{R}{\pi}
\end{aligned}\right\}\ldots\ldots(19),$$

Again, if x denote any circle, we have, from (1),

$$\left.\begin{aligned}
\pi_{x,A} &= \frac{c^2}{\pi_2}\pi_{x,2} + \frac{b^2}{\pi_3}\pi_{x,3} - \frac{t_1^2}{\pi}\pi_{x,s} \\
\pi_{x,B} &= \frac{c^2}{\pi_1}\pi_{x,1} + \frac{a^2}{\pi_3}\pi_{x,3} + \frac{t_2^2}{\pi}\pi_{x,s} \\
\pi_{x,C} &= \frac{b^2}{\pi_1}\pi_{x,1} + \frac{a^2}{\pi_2}\pi_{x,2} + \frac{t_3^2}{\pi}\pi_{x,s} \\
\pi_x &= \frac{t_1^2}{\pi_1}\pi_{x,1} + \frac{t_2^2}{\pi_2}\pi_{x,2} + \frac{t_3^2}{\pi_3}\pi_{x,3} - \frac{2r^2}{\pi}\pi_{x,s}
\end{aligned}\right\} \dots\dots (20).$$

where $\pi_{x,s}$, π_x denotes the powers of (x) with respect to the circum-circle and orthogonal circle.

And again, from (1),

$$\left.\begin{aligned}
\frac{\pi_{x,1}}{2r_1} &= -\frac{r_1}{\pi_1}\pi_{x,A} + \frac{r_2\cos\gamma}{\pi_2}\pi_{x,B} \\
&\qquad + \frac{r_3\cos\beta}{\pi_3}\pi_{x,C} + \frac{R\sin(s-\alpha)}{\pi}\pi_x \\
\frac{\pi_{x,2}}{2r_2} &= \frac{r_1\cos\gamma}{\pi_1}\pi_{x,A} - \frac{r_2}{\pi_2}\pi_{x,B} \\
&\qquad + \frac{r_3\cos\alpha}{\pi_3}\pi_{x,C} + \frac{R\sin(s-\beta)}{\pi}\pi_x \\
\frac{\pi_{x,3}}{2r_3} &= \frac{r_1\cos\beta}{\pi_1}\pi_{x,A} + \frac{r_2\cos\alpha}{\pi_2}\pi_{x,B} \\
&\qquad - \frac{r_3}{\pi_3}\pi_{x,C} + \frac{R\sin(s-\gamma)}{\pi}\pi_x \\
\frac{\pi_{x,s}}{2R} &= \frac{r_1\sin(s-\alpha)}{\pi_1}\pi_{x,A} + \frac{r_2\sin(s-\beta)}{\pi_2}\pi_{x,B} \\
&\qquad + \frac{r_3\sin(s-\gamma)}{\pi_3}\pi_{x,C} - \frac{R}{\pi}\pi_x
\end{aligned}\right\} \dots\dots (21).$$

14. Again, from (8) we have,

$$-16\,.\,\Delta(ABC)\,\Delta(FGH) = \frac{\pi_1\pi_2\pi_3}{\pi}.$$

But $\quad \Delta(ABC) = \dfrac{abc}{4R}, \qquad \Delta(FGH) = \dfrac{r_1r_2r_3}{r}\,S^{\frac{1}{2}}.$

Hence $$\frac{Rr}{\pi} = -4\frac{ar_1}{\pi_1}\,\frac{br_2}{\pi_2}\,\frac{cr_3}{\pi_3} \dots\dots\dots\dots\dots\dots (22).$$

Also, from § 8, we see that

$$\frac{\text{area of triangle } ABC}{\text{area of triangle } FGH} = \frac{\text{area of } EBC}{\text{area of } OGH} =, \&\text{c.}$$

15. If $A'B'C'$ be the other three points in which the given three circles intersect, so that A', B', C' are the inverse points of A, B, C with respect to the orthogonal circle; we may deduce from § 10 some remarkable theorems.

Thus let π'_1, a', t'_1, &c., denote elements for this triangle corresponding to π_1, a, t_1, &c., for the triangle ABC; by (12) we have

$$\frac{1}{\pi_1}+\frac{1}{\pi'_1}+\frac{1}{\pi_2}+\frac{1}{\pi'_2}+\frac{1}{\pi_3}+\frac{1}{\pi'_3}=\frac{1}{r^2}\text{ ....(23)};$$

that is, the sum of the reciprocals of the powers of the six points in which three circles intersect with respect to these circles is equal to the reciprocal of the square of the radius of the circle which cuts the given circles orthogonally.

Again, by (13), we have

$$\frac{t_1^2}{\pi_1}+\frac{t_1'^2}{\pi'_1}+\frac{t_2^2}{\pi_2}+\frac{t_2'^2}{\pi'_2}+\frac{t_3^2}{\pi_3}+\frac{t_3'^2}{\pi'_3}=0\text{ ....(24)},$$

where t_1, t'_1, &c., are the tangents from the six points to the orthogonal circle.

Again, we have, from (18),

$$1=\frac{c^2}{\pi_2}+\frac{b^2}{\pi_3}+\frac{t_1^2}{\pi},$$

and

$$1=\frac{c'^2}{\pi'_2}+\frac{b'^2}{\pi'_3}+\frac{t_1'^2}{\pi'}.$$

But

$$-\frac{1}{r^2}=\frac{1}{\pi}+\frac{1}{\pi'}=\frac{1}{t_1^2}+\frac{1}{t_1'^2}.$$

Hence

$$\frac{c^2}{\pi_2 t_1^2}+\frac{b^2}{\pi_3 t_1^2}+\frac{c'^2}{\pi'_2 t_1'^2}+\frac{b'^2}{\pi'_3 t_1'^2}=0\text{ ....(25)},$$

with two similar formulæ.

16. Solving (19) we deduce at once that

$$\left.\begin{array}{llllll}
4S\sec s\,\dfrac{R}{\pi} = & \dfrac{\sin\alpha}{r_1} & +\dfrac{\sin\beta}{r_2} & +\dfrac{\sin\gamma}{r_3} & -\dfrac{2\cos s}{R} \\
4S\sec s\,\dfrac{r_1}{\pi_1} = & & \dfrac{\cos(s-\gamma)}{r_2} & +\dfrac{\cos(s-\beta)}{r_3} & +\dfrac{\sin\alpha}{R} \\
4S\sec s\,\dfrac{r_2}{\pi_2} = & \dfrac{\cos(s-\gamma)}{r_1} & & \dfrac{\cos(s-a)}{r_3} & +\dfrac{\sin\beta}{R} \\
4S\sec s\,\dfrac{r_3}{\pi_3} = & \dfrac{\cos(s-\beta)}{r_1} & +\dfrac{\cos(s-\alpha)}{r_2} & & +\dfrac{\sin\gamma}{R}
\end{array}\right\}\text{...(26).}$$

From the first of these we have, since

$$\pi = 2Rr\cos\omega = -2Rr\,.\,S^{\frac{1}{2}}\sec s,$$

$$\frac{2\cos s}{R} = \frac{2S^{\frac{1}{2}}}{r} + \frac{\sin\alpha}{r_1} + \frac{\sin\beta}{r_2} + \frac{\sin\gamma}{r_3},$$

which may be written

$$\frac{\cos s}{R} = \frac{1}{r_1 r_2 r_3}(\Delta + \Delta_1 + \Delta_2 + \Delta_3),$$

where Δ, Δ_1, Δ_2, Δ_3 denote the areas of the triangles FGH, AGH, BHF, CFG respectively. This formula is to be found in § 37 of the paper already cited (see *Quarterly Journal*, vol. XXI, p. 46); as is also the following, which is easily deduced from (26):

$$\frac{\cos s}{RR'} = \frac{\cos(s-\alpha)}{r_2 r_3} + \frac{\cos(s-\beta)}{r_3 r_1} + \frac{\cos(s-\gamma)}{r_1 r_2} \quad \dots\dots(27).$$

17. Solving equations (21), we obtain a similar set of formulæ:—

$$\left.\begin{aligned}
4S\sec s\,\frac{R}{\pi}\,\pi_x &= \frac{\sin\alpha}{r_1}\pi_{x,1} + \frac{\sin\beta}{r_2}\pi_{x,2} \\
&\qquad + \frac{\sin\gamma}{r_3}\pi_{x,3} - \frac{2\cos s}{R}\pi_{x,s} \\
4S\sec s\,\frac{r_1}{\pi_1}\,\pi_{x,A} &= \frac{\cos(s-\gamma)}{r_2}\,\pi_{x,2} \\
&\qquad + \frac{\cos(s-\beta)}{r_3}\pi_{x,3} + \frac{\sin\alpha}{R}\pi_{x,s} \\
4S\sec s\,\frac{r_2}{\pi_2}\,\pi_{x,B} &= \frac{\cos(s-\gamma)}{r_1}\,\pi_{x,1} \\
&\qquad + \frac{\cos(s-\alpha)}{r_3}\pi_{x,3} + \frac{\sin\beta}{R}\pi_{x,s} \\
4S\sec s\,\frac{r_3}{\pi_3}\,\pi_{x,C} &= \frac{\cos(s-\beta)}{r_1}\,\pi_{x,1} \\
&\qquad + \frac{\cos(s-\alpha)}{r_2}\pi_{x,2} + \frac{\sin\gamma}{R}\pi_{x,s}
\end{aligned}\right\} \dots\dots(28).$$

The first of these may be written in the form

$$2\,\frac{S\sec s\cos\varpi}{\cos\omega} = \sin\alpha\cos\theta + \sin\beta\cos\phi + \sin\gamma\cos\psi - 2\cos s\cos\chi,$$

θ, ϕ, ψ, χ, ϖ being the angles at which any circle cuts the three given circles, the circum-circle, and the orthogonal circle.

18. Comparing this set of equations with the set (20), we deduce at once that

$$\left.\begin{aligned} 4S\sec s &= \frac{\pi^2}{R^2r^2}\cos s \\ &= \frac{\pi\pi_1}{Rr_1t_1^{\,2}}\sin\alpha = \frac{\pi\pi_2}{Rr_2t_2^{\,2}}\sin\beta = \frac{\pi\pi_3}{Rr_3t_3^{\,2}}\sin\gamma \\ &= \frac{\pi_2\pi_3}{r_2r_3a^2}\cos(s-\alpha) = \frac{\pi_3\pi_1}{r_3r_1b^2}\cos(s-\beta) = \frac{\pi_1\pi_2}{r_1r_2c^2}\cos(s-\gamma) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(29).$$

From the first of these we have

$$\cos\omega = \pm S^{\frac{1}{2}}\sec s.$$

We also have from (29)

$$\frac{a^2t_1^{\,2}}{\sin\alpha\cos(s-\alpha)} = \frac{b^2t_2^{\,2}}{\sin\beta\cos(s-\beta)} = \frac{c^2t_3^{\,2}}{\sin\gamma\cos(s-\gamma)}$$

$$= 16S^2\sec^2 s\,\frac{\pi\pi_1\pi_2\pi_3}{Rr_1r_2r_3}\ldots(30),$$

a formula which is analogous to the formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

for a rectilinear triangle.

We may also deduce from (29) many other relations, for instance, by (27), we have

$$a^2\pi_1 + b^2\pi_2 + c^2\pi_3 = \frac{\pi_1\pi_2\pi_3\cos^2 s}{4S\,RR'}$$

Hence

$$\frac{a^2\pi_1 + b^2\pi_2 + c^2\pi_3}{\pi_1\pi_2\pi_3} = \frac{a'^2\pi'_1 + b'^2\pi'_2 + c'^2\pi'_3}{\pi'_1\pi'_2\pi'_3}.$$

19. We may also notice that if ξ, η, ζ denote the areal coordinates of E, the centre of the orthogonal circle, referred to the triangle ABC; or of O the centre of the circum-circle referred to the triangle FGH: then since by § 7,

$$\pi_1\xi = \pi_2\eta = \pi_3\zeta$$

we have by (29)

$$\frac{\xi}{\frac{a^2 \sec(s-\alpha)}{r_1}} = \frac{\eta}{\frac{b^2 \sec(s-\beta)}{r_2}} = \frac{\zeta}{\frac{c^2 \sec(s-\gamma)}{r_3}} \ldots(31).$$

20. If as in fig. (6) *ABC* lie within the triangle *FGH*; α, β, γ the angles of the circular triangle *ABC* are the angles of intersection of the given circles: in this case π, t_1^2, t_2^2, t_3^2 are all negative. The triangle *A′B′C′* is the inverse of the triangle *ABC* with respect to the orthogonal circle, and the angles of this triangle are $2\pi-\alpha$, $2\pi-\beta$, $2\pi-\gamma$: but in this case R' the radius of the circle *A′B′C′* is to be taken with the negative sign (see *Quarterly Journal*, vol. XXI. page 9, note).

Let us suppose now that the points *ABC* coincide, in this case $2s=\alpha+\beta+\gamma=\pi$: and regarding *A′B′C′* as our triangle; t_1, t_2, t_3 are the distances of the points from *A′*, *B′*, *C′*.

21. Thus we see that *ABC* being any triangle, and *P* any point within it; then if the circles *BPC*, *CPA*, *APC* intersect at angles α, β, γ: and π_1, π_2, π_3 denote the powers of *A*, *B*, *C* with respect to these circles; r_1, r_2 r_3 denote their radii, and x, y, z the distances *AP*, *BP*, *CP*; and π denote the power of *P* with respect to the circum-circle *ABC*; we have by (29).

$$4\sin\alpha\sin\beta\sin\gamma = -\frac{\pi\pi_1\sin\alpha}{Rr_1x^2} = -\frac{\pi\pi_2\sin\beta}{Rr_3y^2} = -\frac{\pi\pi_3\sin\gamma}{Rr_2z^2}$$

$$= \frac{\pi_2\pi_3\sin\alpha}{r_2r_3a^2} = \frac{\pi_3\pi_1\sin\beta}{r_3r_1b^2} = \frac{\pi_1\pi_2\sin\gamma}{r_1r_2c^2}$$

Hence

$$\frac{ax}{\sin\alpha} = \frac{by}{\sin\beta} = \frac{cz}{\sin\gamma} = 2R\frac{xyz}{-\pi};$$

and

$$\frac{\pi_1\pi_2\pi_3}{r_1r_2r_3} = 8abc\sin\alpha\sin\beta\sin\gamma.$$

MATHEMATICAL NOTES.

Proof of the formula for the torsion of a geodesic.

Let $OP=ds$ be an element of a geodesic; ON, NP the resolved parts of OP parallel to the two lines of curvature; θ the angle PON. The angle of torsion $d\tau$ measures the rotation of the normal to the surface round OP. But this is equal to the sum of the rotations round ON and NP resolved along OP. For in travelling along ON and NP, the normal rotates only about NP and ON respectively. Therefore if ω_1, ω_2 be the rotations round ON, NP,

$$d\tau = \omega_1 \cos\theta + \omega_2 \sin\theta.$$

But
$$\omega_1 = -\frac{ds \sin\theta}{\rho_2}, \quad \omega_2 = \frac{ds\cos\theta}{\rho_1},$$

ρ_1, ρ_2 being the principal radii of curvature of the surface. Therefore if σ be the radius of torsion

$$\frac{1}{\sigma} = \frac{d\tau}{ds} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\theta\cos\theta.$$

G. S. Carr.

A proof of Holditch's Theorem.

Let CC' be a line of given length, moving with its extremities on two fixed closed curves; P a given point on the line, dividing it into parts $CP=c$, $PC'=c'$. Let (C), (P), (C') denote the areas of the curves traced in a complete revolution by C, P and C' respectively. Let O be the point of intersection of the directions of the line in two contiguous positions inclined to each other at an angle $d\psi$.

Then, by a known theorem, if we join any point A to C, P, and C',

$$AC^2.c' + AC'^2.c = AP^2.(c+c') + cc'(c+c').$$

Making A coincide with O, we have

$$OC^2.c' + OC^2.c = OP^2(c+c') + cc'(c+c').$$

But $\frac{1}{2}OC^2.d\psi$ is an element of the difference of the curves traced by C and O: similarly for $\frac{1}{2}OP^2.d\psi$, and for $\frac{1}{2}OC'^2.d\psi$.

Hence, integrating for a complete revolution

$$2c'.\{(C)-(O)\}+2c\{(C')-(O)\}$$
$$=2(c+c')\{(P)-(O)\}+2\pi cc'(c+c')$$

whence

$$c'.(C)+c(C')=(c+c')(P)+\pi cc'(c+c').$$

M. JENKINS.

Correction to a previous paper.

In my paper on Cayley's Differential Equation for orthognal surfaces (*antè*, pp. 27–33), the direction cosines of the principal tangents on the consecutive surface should have been taken as

$$\alpha_1+\alpha_2\,\partial\theta-\frac{l\partial\nu}{\sigma}\sin i,\ \beta_1+\beta_2\,\partial\theta-\frac{m\partial\nu}{\sigma}\sin i,\ \text{etc.};$$

$$\alpha_2-\alpha_1\,\partial\theta-\frac{l\partial\nu}{\sigma}\cos i,\ \beta_2-\beta_1\,\partial\theta-\frac{m\partial\nu}{\sigma}\cos i,\ \text{etc.},$$

σ denoting the radius of curvature of the trajectory, and i the angle between the binormal to the trajectory and the principal tangent α, β, γ. The resulting equation (B) should have its left-hand member multiplied by 4, and the first three of formulæ (C) should be changed to

$$\tfrac{1}{2}\mathfrak{A}=L^2P(M^2-N^2)+(N^2R-M^2Q)(M^2+N^2),\ \text{etc.}$$

A. R. JOHNSON.

NOTE ON TRIPLE ALGEBRA.

By *Arthur Buchheim, M.A.*

IN a note on double algebra, published in a recent number of the *Messenger of Mathematics* (*ante* p. 62), I solved the problem of finding pairs of matrices that could be taken as the units of a double algebra. In the present note I consider a particular case of the same problem for triple algebra. It is convenient in classifying multiple algebras formed from matrices to speak of n-triple m-ary algebras, meaning algebras derived from n units being m-ary matrices.

In the note referred to above I proved that if a double binary algebra does not contain the matrix unity all its matrices must be vacuous. The proof given there is obviously quite general, and the theorem is true for any multiple algebra: it is convenient to describe an algebra* as vacuous when it consists entirely of vacuous matrices. Moreover, it is easy to show that the matrices of any vacuous algebra form themselves into sub-groups, all the matrices of a sub-group having a null point in common, and constituting an algebra by themselves: for want of a better name we may call the algebra formed by the matrices of a sub-group a connected algebra.

Lastly, unless all the matrices of connected n-triple n-ary algebra satisfy equations of the form $x^r = 0$, $r < n$, we can always (as is not hard to see by counting constants and equations) get matrices such that their nth power vanishes and no lower power vanishes: in this case we can call the connected algebra a general connected algebra.

In this note I consider general connected ternary triple algebras.

Let x be a ternary matrix such that $x^3 = 0$, and y another matrix: we take x, x^2, y as units, and we have to see how to determine y: let e_1, e_2, e_3 be the latent points of x, and let e_1 be a null point of y: let the canonical form of x be

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and let y be

$$\begin{pmatrix} 0 & a & a' \\ 0 & b & b' \\ 0 & c & c' \end{pmatrix}.$$

The canonical form of x^2 is, of course,

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and it is obvious that $yx^2 = 0$.

---

* When I speak of an algebra I mean what is more fully defined as a linear associative algebra.

Now let $$xy = \alpha x + \beta x^2 + \gamma y,$$
then $$0 = x^3 y = \gamma x^2 y,$$
that is, either γ or $x^2 y$ must vanish.

Consider, first the case in which γ vanishes, we are to have

$$xy = \alpha x + \beta x^2.$$

Now
$$xye_1 = be_1 + ce_2,{}^{*}$$
$$\alpha xe_1 + \beta x^2 e_1 = \alpha e_1,$$
$$xye_3 = b'e_1 + c'e_2,$$
$$\alpha xe_3 + \beta x^2 e_3 = \alpha e_2 + \beta e_1;$$
therefore
$$\alpha = c' = b,$$
$$\beta = b',$$
$$c = 0.$$

We have therefore reduced y to the form

$$\begin{array}{ccc} (\,0, & a, & a'\,). \\ \left|\,0,\right. & b, & \left.b'\,\right| \\ \left|\,0,\right. & 0, & \left.b\,\right| \end{array}$$

Now consider yx; we have

$$yx = \alpha x + \beta x^2 + \gamma y,$$
$$0 = yx^2 = \alpha x^2 + \gamma yx,$$
now
$$yxe_2 = ye_1 = 0,$$
$$yxe_3 = ye_2 = ae_1 + be_2,$$
$$\alpha x^2 e_2 = 0,$$
$$\alpha x^2 e_3 = \alpha e_1;$$
we must therefore have
$$\alpha = a,$$
$$b = 0.$$

---

[*] Of course we can, if we choose, multiply the matrices in the usual way, but I find this way more convenient.

The form of y is now

$$\begin{array}{c}(\;0,\quad a,\quad a'\;),\\ \left|\begin{array}{ccc}0, & 0, & b'\\ 0, & 0, & 0\end{array}\right|\end{array}$$

and it will be found that

$$y^2 = b'x^2,$$

and that $x^2y = 0$.

The algebra can be derived from the three matrices

$$\begin{array}{c}(\;0,\quad 1,\quad 0\;),\\ \left|\begin{array}{ccc}0, & 0, & 0\\ 0, & 0, & 0\end{array}\right|\end{array}\qquad \begin{array}{c}(\;0,\quad 0,\quad 1\;),\\ \left|\begin{array}{ccc}0, & 0, & 0\\ 0, & 0, & 0\end{array}\right|\end{array}\qquad \begin{array}{c}(\;0,\quad 0,\quad 0\;).\\ \left|\begin{array}{ccc}0, & 0, & 1\\ 0. & 0, & 0\end{array}\right|\end{array}$$

It will be observed, that the third of these satisfies $x^2 = 0$.

We have now to consider the case in which x^2y vanishes; it is easy to see that c, c' must both vanish, so that the form of y is

$$\begin{array}{c}(\;0,\quad a,\quad a'\;).\\ \left|\begin{array}{ccc}0, & b, & b'\\ 0, & 0, & 0\end{array}\right|\end{array}$$

Let $$xy = \alpha x + \beta x^2 + \gamma y,$$

then $$0 = x^2y = \alpha x^2 + \gamma xy.$$

Now $$xye_2 = be_1,$$

$$xye_3 = b'e_1,$$

$$x^2e_2 = 0,$$

$$x^2e_3 = \alpha e_1,$$

we must therefore have $b = 0$, so that the form of y is

$$\begin{array}{c}(\;0,\quad a,\quad a'\;),\\ \left|\begin{array}{ccc}0, & 0, & b'\\ 0, & 0, & 0\end{array}\right|\end{array}$$

and we get the same algebra as before.

The Grammar School, Manchester,
August 30, 1886.

THE PERIOD EQUATION FOR LATERAL VIBRATIONS.

By *A. G. Greenhill, M.A.*

1. In the discussion of the Lateral Vibrations of a uniform elastic bar, the ends of which are either clamped or free, the solution of the transcendental equations

$$\cos m \cosh m = 1 \text{..................}(1),$$

and

$$\cos m \cosh m = -1 \text{..................}(2),$$

is required (Rayleigh, *Theory of Sound*, I., § 174, p. 222).

With Cayley's notation, equation (1) is equivalent to

$$m = \operatorname{gd} m,$$

or

$$m = \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}m).$$

But the substitutions of

$$\cos m = \frac{1 - \tan^2 \frac{1}{2}m}{1 + \tan^2 \frac{1}{2}m},$$

$$\cosh m = \frac{1 + \tanh^2 \frac{1}{2}m}{1 - \tanh^2 \frac{1}{2}m},$$

reduce these equations (1) and (2) to the more symmetrical form

$$\tanh^2 \tfrac{1}{2}m = \tan^2 \tfrac{1}{2}m,$$

$$\tanh^2 \tfrac{1}{2}m = \cot^2 \tfrac{1}{2}m,$$

or

$$\tanh \tfrac{1}{2}m = \pm \tan \tfrac{1}{2}m \text{..................}(3),$$

$$\tanh \tfrac{1}{2}m = \mp \cot \tfrac{1}{2}m \text{..................}(4);$$

the upper sign giving the even and the lower the odd series of vibrations.

2. The general differential equation of the lateral vibrations, neglecting rotatory inertia of cross section, of a uniform elastic straight bar of density ρ, cross section A, and flexural rigidity EAk^2, where E denotes Young's modulus, and Ak^2 the moment of inertia of the cross section of the bar about an

axis through the centre of area of the section perpendicular to the plane of flexure is

$$\rho A \frac{d^2u}{dt^2} = -EAk^2 \frac{d^4u}{dx^4},$$

(Rayleigh, *Sound*, I., § 170), where u denotes the displacement of the mean fibre; and supposing p the number of vibrations per second, so that u is proportional to $\sin 2\pi pt$ or $\cos 2\pi pt$, then

$$Ek^2 \frac{d^4u}{dx^4} = 4\pi^2 p^2 \rho u,$$

or

$$\frac{d^4u}{dx^4} = \frac{m^4}{l^4} u,$$

where

$$\frac{m^4}{l^4} = \frac{4\pi^2 p^2 \rho}{Ek^2},$$

or

$$p = \frac{m^2 k}{2\pi l^2} \sqrt{\frac{E}{\rho}},$$

and l denotes the length of the bar.

Also if the bar is revolving about its axis, then p will be the number of revolutions per second at which the straight form of the bar becomes unstable, and the bar begins to curve under centrifugal whirling.

3. Then, taking the origin at the middle point of the bar, the solution, for a free-free bar, of the differential equation

$$\frac{d^4u}{dx^4} = \frac{m^4}{l^4} u,$$

is (i) for the even vibrations,

$$u = A \sin mx/l + B \sinh mx/l,$$

subject to the conditions

$$\frac{d^2u}{dx^2} = 0, \text{ and } \frac{d^3u}{dx^3} = 0,$$

at the ends, where $x = \pm \frac{1}{2} l$; so that

$$-A \sin \tfrac{1}{2}m + B \sinh \tfrac{1}{2}m = 0,$$

$$-A \cos \tfrac{1}{2}m + B \cosh \tfrac{1}{2}m = 0;$$

or

$$\frac{A}{B} = \frac{\sinh \frac{1}{2}m}{\sin \frac{1}{2}m} = \frac{\cosh \frac{1}{2}m}{\cos \frac{1}{2}m};$$

so that

$$\tanh \tfrac{1}{2}m = \tan \tfrac{1}{2}m;$$

and (ii) for the odd vibrations,

$$u = A \cos mx/l + B \cosh mx/l,$$

subject to the conditions

$$\frac{d^2u}{dx^2} = 0, \text{ and } \frac{d^3u}{dx^3} = 0,$$

at the ends, where $x = \pm \frac{1}{2}l$; so that

$$-A \cos \tfrac{1}{2}m + B \cosh \tfrac{1}{2}m = 0,$$

$$A \sin \tfrac{1}{2}m + B \sinh \tfrac{1}{2}m = 0;$$

or $$\frac{A}{B} = \frac{\cosh \frac{1}{2}m}{\cos \frac{1}{2}m} = -\frac{\sinh \frac{1}{2}m}{\sin \frac{1}{2}m};$$

so that $$\tanh \tfrac{1}{2}m = -\tan \tfrac{1}{2}m.$$

4. In a clamped-clamped bar, the even and odd vibrations are given as before, by

$$u = A \sin mx/l + B \sinh mx/l,$$

$$u = A \cos mx/l + B \cosh mx/l;$$

but now, when $x = \pm \frac{1}{2}l$,

$$u = 0, \text{ and } \frac{du}{dx} = 0;$$

so that, for the even vibrations,

$$A \sin \tfrac{1}{2}m + B \sinh \tfrac{1}{2}m = 0,$$

$$A \cos \tfrac{1}{2}m + B \cosh \tfrac{1}{2}m = 0;$$

or $$\frac{A}{B} = -\frac{\cosh \frac{1}{2}m}{\sin \frac{1}{2}m} = -\frac{\cosh \frac{1}{2}m}{\cos \frac{1}{2}m},$$

so that $$\tanh \tfrac{1}{2}m = \tan \tfrac{1}{2}m;$$

and, for the odd vibrations,

$$A \cos \tfrac{1}{2}m + B \cosh \tfrac{1}{2}m = 0,$$

$$-A \sin \tfrac{1}{2}m + B \sinh \tfrac{1}{2}m = 0;$$

or $$\frac{A}{B} = -\frac{\cosh \frac{1}{2}m}{\cos \frac{1}{2}m} = \frac{\sinh \frac{1}{2}m}{\sin \frac{1}{2}m},$$

so that $$\tanh \tfrac{1}{2}m = -\tan \tfrac{1}{2}m.$$

5. In a clamped-free bar, clamped at $x = -\frac{1}{2}l$, and free at $x = \frac{1}{2}l$, we must put

$$u = A \cos mx/l + B \sinh mx/l,$$

or

$$u = A \sin mx/l + B \cosh mx/l;$$

for the even and odd vibrations respectively, and then the conditions that

$$u = 0, \quad \frac{du}{dx} = 0, \quad \text{when } x = -\tfrac{1}{2}l,$$

$$\frac{d^2u}{dx^2} = 0, \quad \frac{d^3u}{dx^3} = 0, \quad \text{when } x = \tfrac{1}{2}l,$$

are satisfied, provided that

$$\tanh \tfrac{1}{2}m = -\cot \tfrac{1}{2}m,$$

or

$$\tanh \tfrac{1}{2}m = \cot \tfrac{1}{2}m.$$

6. Denoting by $m_0, m_1, m_2, m_3, \ldots$ the positive roots of the equation

$$\cos m \cosh m = 1;$$

and putting $\quad m_i = (i + \frac{1}{2})\pi - (-1)^i \beta_i,$

(Rayleigh, *Sound*, I, § 174), where i denotes a positive integer, then β becomes smaller as i is increased; also $\beta_0 = \frac{1}{2}\pi$, since $m_0 = 0$.

Then $\quad m_0, m_2, m_4, m_6, \ldots$

are the roots of $\quad \tanh \frac{1}{2}m = \tan \frac{1}{2}m,$

and $\quad m_1, m_3, m_5, m_7, \ldots$

are the roots of $\quad \tanh \frac{1}{2}m = -\tan \frac{1}{2}m.$

7. Again, denoting by $n_1, n_2, n_3, n_4, \ldots$ the positive roots of the equation

$$\cos n \cosh n = -1,$$

and putting $\quad n_i = (i - \frac{1}{2})\pi - (-1)^i \alpha_i$

(Rayleigh, *Sound*, I. § 175), where i denotes a positive integer, then here also α_i becomes smaller as i is increased.

Then $n_2, n_4, n_6, \ldots$ are the roots of

$$\tanh \tfrac{1}{2}n = -\cot \tfrac{1}{2}n;$$

and $n_1, n_3, n_5, \ldots$ are the roots of

$$\tanh \tfrac{1}{2}n = \cot \tfrac{1}{2}n.$$

8. A geometrical illustration is afforded by drawing the curve

$$y = \tanh \tfrac{1}{2}x$$

for positive values of x; and also the curves

$$y = \pm \tan \tfrac{1}{2}x,$$

and
$$y = \pm \cot \tfrac{1}{2}x,$$

for positive values of x, limited between $y = 0$ and $y = 1$.

Then the points of intersection of these curves will indicate to the eye the values of m_i and n_i (fig. 7), and incidentally will show that it is not possible for the equation

$$\alpha_{i+1} = \beta_i$$

to hold, as asserted in Rayleigh's *Sound*, I., § 174.

Denoting $e^{(i+\frac{1}{2})\pi}$ by a, and taking the expression for β_i given there in a series of descending forms of a,

$$\beta_i = \frac{2}{a} + (-1)^i \frac{4}{a^2} + \frac{34}{3a^3} + (-1)^i \frac{112}{3a^4} + \dots,$$

then
$$\alpha_{i+1} = \frac{2}{a} - (-1)^i \frac{4}{a^2} + \frac{34}{3a^3} - (-1)^i \frac{112}{3a^4} + \dots;$$

and
$$\alpha_1 = \cdot 3043077,$$

$$\alpha_2 = 10^{-1} \times \cdot 182974, \qquad \beta_1 = 10^{-1} \times \cdot 176518,$$

$$\alpha_3 = 10^{-3} \times \cdot 768894, \qquad \beta_2 = 10^{-3} \times \cdot 777010,$$

$$\alpha_4 = 10^{-4} \times \cdot 335507, \qquad \beta_3 = 10^{-4} \times \cdot 335505,$$

$$\alpha_5 = 10^{-5} \times \cdot 144989, \qquad \beta_4 = 10^{-5} \times \cdot 144989.$$

9. Another geometrical interpretation can be given, in connexion with the circle

$$x^2 + y^2 = 1,$$

and the rectangular hyperbola

$$x^2 - y^2 = 1.$$

Denoting the sectorial area OAP of the hyperbola (fig. 8) by $\frac{1}{2}m$, then it is known that

$$OM = \cosh m, \qquad MP = \sinh m.$$

Also, if MQ is drawn to touch the circle at Q, and the angle AOQ is denoted by θ, and therefore the sectorial area OAQ by $\frac{1}{2}\theta$, then

$$ON = \cos\theta = \operatorname{sech} m,$$

$$NQ = \sin\theta = \tanh m,$$

and
$$OM = \cosh m = \sec\theta,$$

$$MP = \sinh m = \tan\theta,$$

or, in Cayley's notation,

$$\theta = \operatorname{gd} m,$$

equivalent to
$$m = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta),$$

or
$$\tanh^2 \tfrac{1}{2}m = \tan^2 \tfrac{1}{2}\theta,$$

so that if MQ meets the common tangent at A in T, then the line OT, which bisects the angle AOQ, bisects also the sectorial area OAP, and NTP is the tangent at P to the hyperbola; also PQ is parallel to OT, and passes through the other vertex of the hyperbola; and PA produced passes through Q', the point opposite to Q on the circle.

10. Let OT meet the hyperbola in p and the circle in q. Then if m_{2i} denotes a root of the equation

$$\tanh \tfrac{1}{2}m = \tan \tfrac{1}{2}m,$$

we have, since
$$m_{2i} = 2i\pi + \tfrac{1}{2}\pi - \beta_{2i},$$
in consequence

$$2 \text{ area } AOp = i\pi + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta_{2i},$$

$$2 \text{ area } AOq = \tfrac{1}{4}\pi - \tfrac{1}{2}\beta_{2i};$$

so that
$$2 \text{ area } Apq = i\pi.$$

Also, if n_{2i} denotes a root of the equation

$$\tanh \tfrac{1}{2}n = -\cot \tfrac{1}{2}n;$$

then we shall have, in a similar manner,

$$2 \text{ area } Apq = (i - \tfrac{1}{2})\pi;$$

so that the area Apq

$$= \tfrac{1}{2}\pi,\ \pi,\ \tfrac{3}{2}\pi,\ 2\pi,\ \dots \text{ for } m_2,\ m_4,\ m_6,\ m_8,\ \dots,$$

and
$$= \tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi,\ \tfrac{5}{4}\pi,\ \tfrac{7}{4}\pi,\ \dots \text{ for } n_2,\ n_4,\ n_6,\ n_8,\ \dots.$$

11. Again, if m_{2i-1} denotes a root of

$$\tanh \tfrac{1}{2}m = -\tan \tfrac{1}{2}m,$$

then, since
$$m_{2i-1} = 2i\pi - \tfrac{1}{2}\pi + \beta_{2i-1},$$

in the corresponding geometrical interpretation

$$2 \text{ area } AOp = i\pi - \tfrac{1}{4}\pi + \tfrac{1}{2}\beta_{2i-1},$$

$$2 \text{ area } AOq = \tfrac{1}{4}\pi - \tfrac{1}{2}\beta_{2i-1};$$

so that

$$\text{area } AOp + \text{area } AOq = \tfrac{1}{2}i\pi.$$

Similarly, in the geometrical interpretation of the roots n_{2i+1} of

$$\tanh \tfrac{1}{2}n = \cot \tfrac{1}{2}n,$$

we shall find

$$2 \text{ area } AOp = i\pi + \tfrac{1}{4}\pi + \tfrac{1}{2}\alpha_{2i+1},$$

$$2 \text{ area } AOq = \tfrac{1}{4}\pi - \tfrac{1}{2}\alpha_{2i+1};$$

and therefore

$$\text{area } AOp + \text{area } AOq = \tfrac{1}{2}(i + \tfrac{1}{2})\pi.$$

12. The same period equations are required in the determination of standing waves across a canal, the sides of which slope at 45° to the horizon.

Taking the axis of x along the bottom edge of the canal, and the axis of z vertically upwards; then the velocity function, omitting a small constant factor A, and a periodic factor $\cos nt$,

$$\phi = \sin py \sinh pz + \sinh py \sin pz,$$

or

$$\phi = \cos py \cosh pz + \cosh py \cos pz,$$

(Kirchhoff, *Gesammelte Abhandlungen*, II, p. 440), satisfies the equation of continuity

$$\frac{\delta^2\phi}{\delta y^2} + \frac{\delta^2\phi}{\delta z^2} = 0,$$

and also the boundary conditions at the sides of the canal, viz.

$$\text{(i)}\quad \frac{\delta\phi}{\delta y} = \frac{\delta\phi}{\delta z}, \text{ when } y = z,$$

$$\text{(ii)}\quad \frac{\delta\phi}{\delta y} = -\frac{\delta\phi}{\delta z}, \text{ when } y = -z.$$

At the free surface where the mean value of z is h, the depth of water (fig. 9),

$$l\frac{\delta\phi}{\delta z} = \phi,$$

l denoting the length of the equivalent simple pendulum of the wave motion, so that

$$ln^2 = g;$$

and therefore

$$pl\,(\sin py \cosh ph + \sinh py \cos ph)$$
$$= \sin py \sinh ph + \sinh py \sin pl,$$

or

$$pl\,(\cos py \sinh ph - \cosh py \sin ph)$$
$$= \cos py \cosh ph + \cosh py \cos ph,$$

for all values of y; and therefore either

$$pl = \tanh ph = \tan ph,$$

or

$$pl = \coth ph = -\cot ph.$$

Denoting by η the elevation of the free surface above its mean level, the velocity function being $A\phi \cos nt$, then

$$\frac{d\eta}{dt} = A\,\frac{\delta\phi}{\delta z}\cos nt,$$

and

$$\eta = \frac{A}{n}\,\frac{\delta\phi}{\delta z}\sin nt$$

$$= \frac{pA}{n}(\sin py \cosh ph + \sinh py \cos ph) \sin nt,$$

or

$$= \frac{pA}{n}(\cos py \sinh ph - \cosh py \sin ph) \sin nt,$$

and therefore the surface vibrates in exactly the same manner as the lateral vibrations of an elastic bar.

13. We may even suppose the same liquid motion to take place when the surface is covered with ice of uniform thickness e, superficial density σ, and flexural rigidity $L = \frac{1}{12}e^3E$, E denoting Young's modulus for ice; and then supposing the inertia of the ice concentrated in its middle plane, equivalent to neglecting the *rotatory* inertia of the vibrations, as is customary in the discussion of lateral vibrations (Rayleigh, *Sound*, I, § 173), the equation of motion of the ice becomes

$$\sigma\frac{d^2\eta}{dt^2} = -L\frac{d^4\eta}{dy^4} + \delta p,$$

where δp denotes the periodic part of the liquid pressure just under the ice, and therefore

$$\delta p + g\rho\eta + \rho\frac{\delta\phi}{\delta t} = 0.$$

Therefore, since

$$\frac{d^2\eta}{dt^2} = -n^2\eta, \quad \frac{d^4\eta}{dy^4} = p^4\eta,$$

we have $$\sigma n^2\eta = Lm^4\eta + g\rho\eta + \rho\frac{\delta\phi}{\delta t},$$

$$\left\{\left(l - \frac{\sigma}{\rho}\right)n^2 + \frac{Lm^4}{\rho}\right\}\eta = -\frac{\delta\phi}{\delta t},$$

or $$\left\{\left(l - \frac{\sigma}{\rho}\right)n^2 + \frac{Lm^4}{\rho}\right\}\frac{\delta\phi}{\delta z} = n^2\phi,$$

so that $$p\left(l - \frac{\sigma}{\rho} + \frac{Lm^4}{\rho n^2}\right) = \tanh ph = \tan ph,$$

or $$= \coth ph = -\cot ph;$$

the same conditions as before, but now the length of the equivalent simple pendulum is increased by

$$\frac{\sigma}{\rho} - \frac{Lm^4}{\rho n^2}.$$

We suppose here that the ice is free and not attached to the banks, and that the slight periodic ebb and flow of the water at the break between the ice and the banks does not materially affect the result.

14. For progressive waves in this rectangular channel Kelland has given the solution

$$\phi = A\cosh my \cosh mz \cos\sqrt{2}\,(mx - nt),$$

a velocity function which satisfies the equation of continuity and the boundary conditions; and the free surface condition

$$l\frac{\delta\phi}{\delta z} = \phi,$$

for $z = h$, leads to the equation

$$ml \sinh mh = \cosh mh,$$

or $$ml = \coth mh.$$

Denoting again by η, the slight elevation of the surface due to the wave motion

$$\frac{d\eta}{dt} = \frac{\delta\phi}{\delta z}\,(z = h)$$

$$= mA\cosh my \sinh mh \cos\sqrt{2}\,(mx - nt),$$

and therefore

$$\eta = -\frac{mA}{n\sqrt{2}} \sinh mh \cosh my \sin\sqrt{2}\,(mx - nt),$$

giving the elevation of the surface at any point.

15. Here again we may suppose the surface covered with ice, and investigate the change in the value of l, the length of the equivalent simple pendulum.

The equation of oscillation of the surface is now

$$\sigma \frac{d^2\eta}{dt^2} = -L\left(\frac{\delta^4\eta}{\delta x^4} + 2\,\frac{\delta^4\eta}{\delta x^2\,\delta y^2} + \frac{\delta^2\eta}{\delta y^4}\right) + \delta p,$$

(Thomson and Tait, *Natural Philosophy*, II, § 644),

where $$\delta p + g\rho\eta + \rho\frac{\delta\phi}{\delta t} = 0.$$

Now $$\frac{d^2\eta}{dt^2} = -2n^2\eta,$$

$$\frac{\delta^4\eta}{\delta x^4} + 2\,\frac{\delta^4\eta}{\delta x^2\delta y^2} + \frac{\delta^4\eta}{\delta y^4} = m^4\eta\,;$$

so that $$2\sigma n^2\eta = Lm^4\eta + g\rho\eta + \rho\frac{\delta\phi}{\delta t}$$

$$= Lm^4\eta + g\rho\eta - 2\rho\frac{n^2}{m}\eta \coth mh,$$

or, since $$g = 2ln^2,$$

$$\sigma\frac{g}{l} = Lm^4 + g\rho - \frac{g\rho}{ml}\coth mh,$$

or $$l = \frac{\sigma + \dfrac{\rho}{m}\coth mh}{\rho + \dfrac{Lm^4}{g}},$$

giving l the length of the equivalent simple pendulum when the wave length

$$\lambda = \frac{2\pi}{m\sqrt{2}} = \frac{\pi\sqrt{2}}{m}.$$

Here again the disturbing effect of the discontinuity at the bank is neglected.

16. Another solution, analogous to Kelland's, is obtained by putting

$$\phi = A \sinh my \sinh mz \sin \sqrt{2}\,(mx - nt),$$

satisfying the equation of continuity and the boundary conditions, and now the free surface condition gives

$$ml = \tanh mh,$$

if the surface is free, but if covered with ice, then, as before,

$$l = \frac{\sigma + \frac{\rho}{m} \tanh mh}{\rho + \frac{Lm^4}{g}},$$

where the wave length

$$\lambda = \frac{\pi\sqrt{2}}{m}.$$

THE "SINE-TRIPLE-ANGLE" CIRCLE.

By *R. Tucker, M.A.*

POINTS D, D'; E, E'; F, F' are taken on the sides BC, CA, AB of the triangle ABC, so that

$$\angle AEF = A = \angle AF'E',$$

$$\angle BFD = B = \angle BD'F',$$

$$\angle CDE = C = \angle CE'D',$$

whence we get two inscribed triangles DEF, $D'E'F'$.

Let $\qquad BD = x, \quad CE = y, \quad AF = z;$

$$CD' = x', \quad AE' = y', \quad BF' = z';$$

then

$$x + 2y \cos C = a,$$

$$y + 2z \cos A = b,$$

$$z + 2x \cos B = c,$$

and like equations for x', y', z'.

From these equations we get

$$x\begin{vmatrix}1 & 2\cos C & 0\\ 0 & 1 & 2\cos A\\ 2\cos B & 0 & 1\end{vmatrix} = \begin{vmatrix}a & 2\cos C & 0\\ b & 1 & 2\cos A\\ c & 0 & 1\end{vmatrix}$$

i.e. $x(1+8\cos A\cos B\cos C)=2R\sin(2C-A)=\mu x.$

Similarly $\qquad \mu x'=2R\sin(2B-A).$

The angles of DEF are easily seen to be

$$2B-C,\ 2C-A,\ 2A-B,$$

and of those of $D'E'F'$ to be $2C-B,\ 2A-C,\ 2B-A.$

Hence we can write, if $R=\mu\rho$,

$$x=2\rho\sin E,\quad y=2\rho\sin F,\quad z=2\rho\sin D,$$
$$x'=2\rho\sin F',\ y'=2\rho\sin D',\ z'=2\rho\sin E',$$

and the two triangles have a common circum-circle, whose radius is $=R/\mu$.

Now $\quad \angle DF'D'=\angle BF'D'-\angle BF'D$

$$=(180-2B)-(2C-A)$$
$$=-(180^\circ-3A)=3A-180^\circ$$

$$\angle EFE'=\angle EFD-\angle DFE'$$
$$=(2A-B)-(180-2C)=180^\circ-3B,$$

and $\qquad \angle FDF'=180^\circ-3C;$

hence $\qquad DD' : EE' : FF'$

$$=\sin 3A : \sin 3B : \sin 3C.$$

On inspection it will be seen that

$$FE' : DF' : ED'=\sin A : \sin B : \sin C,$$
$$EF' : FD' : DE'=\sin 2A : \sin 2B : \sin 2C.$$

The equation to the line joining the centre of the circle (O') to the circum-centre (O), is

$$\alpha(b^2-c^2)\sec A+\beta(c^2-a^2)\sec B+\gamma(a^2-b^2)\sec C=0;$$

this line cuts the Symmedian-point axis in a point Q, which may be thus found; project BC orthogonally on AB, AC, and project these lines back on BC, then cut BC inversely in the ratio of these lines in R; AR, and the similar lines through B, C will cointersect in Q.

October 10, 1886.

DEFINITION D'UN INVARIANT.

By Professor *Paul Mansion.*

1. **Lemmes** (sur les coefficients indéterminés). I. *Une fonction algébrique entière de degré m en x est identiquement nulle si elle s'annule pour $m+1$ valeurs distinctes de x.* II. *Une fonction algébrique entière de degré m en x, de degré n en y, est identiquement nulle, si elle s'annule pour $(m+1)$ valeurs de x, $(n+1)$ de y, chacune des valeurs de x étant successivement combinée avec chacune des valeurs de y.* III. *Théorèmes analogues pour les fonctions algébriques entières d'un plus grand nombre de variables.*

Les Lemmes II, III se déduisent du premier; le premier s'établit directement en cherchant, au moyen des déterminants, la valeur des coefficients de la fonction de x.

2. *Lettres formant un groupe indépendant, dans une substitution linéaire.* Considérons des fonctions algébriques homogènes,

$$f_1(x, y, z, \xi, \eta, \zeta, a_1, b_1, c_1, \ldots),$$
$$f_2(x, y, z, \xi, \eta, \zeta, a_2, b_2, c_2, \ldots), \text{ etc.}$$

Elles contiennent plusieurs séries de variables, chaque série renfermant, par exemple, trois variables. Ces variables sont soumises à une même transformation linéaire, savoir:

$$x = l_1x' + m_1y' + n_1z', \quad y = l_2x' + \text{etc.},$$
$$\xi = l_1\xi' + m_1\eta' + n_1\zeta', \quad \eta = l_2\xi' + \text{etc.}$$

Nous disons que

$$(x, y, z), (\xi, \eta, \zeta), (a_1, b_1, c_1, \ldots), (a_2, b_2, c_2, \ldots),$$

forment des groupes de lettres indépendantes si les fonctions sont homogènes par rapport à chacun de ces groupes de variables.

3. Théorème I. *Si les fonctions f_1, f_2, etc., sont telles que, dans certains cas, l'on ait*

$$F(x, y, z, \xi, \eta, \zeta, a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots) = 0 \ldots (1),$$

et si cette propriété subsiste quand les fonctions f_1, f_2, *etc. sont multipliées par une constante arbitraire,* F *est une fonction homogène ou une somme de fonctions homogènes par rapport à chacun des groupes de variables indépendantes.*

En effet, remplacez les lettres de l'un de ces groupes x, y, z, par exemple, par kx, ky, kz. Les fonctions f_1, f_2, etc. seront multipliées par des puissances de k. La propriété (1) subsistera donc et deviendra

$$F(kx, ky, kz, \xi, \eta, \zeta, a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots) = 0.$$

Ou pourra l'écrire

$$F_p k^p + F_q k^q + F_r k^r + \text{etc.} = 0,$$

en désignant par F_p, F_q, F_r, etc. les parties de F qui sont respectivement de degré p, q, r, etc., en x, y, z. On déduit de là, puisque k est quelconque (d'après 1, I)

$$F_p = 0, \; F_q = 0, \; F_r = 0, \text{ etc.}$$

4. Théorème II. *Si la relation*

$$F(x, y, z, \xi, \eta, \zeta, a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots) = 0,$$

entraine la relation

$$F(x', y', z', \xi', \eta', \zeta', a_1, b_1', c_1', \ldots, a_2', b_2', c_2', \ldots) = 0,$$

et reciproquement, les premiers membres de ces équations ne diffèrent que par un facteur fonction seulement de $l, m, n, l_1', m_1', n_1'$, *etc.* (Nous supposons F homogène par rapport à chaque groupe de lettres indépendantes).

Si l'on a

$$F(x', \text{etc.}) = \Delta . F(x, \text{etc.}) \ldots\ldots\ldots\ldots\ (2),$$

Δ ne peut contenir aucune des lettres $x, y, z, \xi, \eta, \zeta, a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots$; car les lettres accentuées x', etc. sont des fonctions linéaires des lettres non accentuées; si donc, on remplace, dans $F(x', \text{etc.})$, les lettres accentueés par leurs valeurs au moyen des lettres non accentuées, le premier membre de la relation (2) sera de même degré que $F(x, \text{etc.})$ par rapport à chaque groupe de lettres indépendantes. Par conséquent, ces lettres ne peuvent entrer dans Δ.

Si l'on n'avait pas identiquement la relation (2), on aurait simultanément

$$F(x', \text{etc.}) = 0, \; F(x, \text{etc.}) = 0,$$

pour *une infinité* de valeurs des lettres qui entrent dans $F(x, \text{etc.})$; par conséquent, pour un nombre de valeurs supérieur au degré de $F(x, \text{etc.})$, par rapport à chacune de ces lettres. Donc (d'après le No. 1, III), la fonction $F(x', \text{etc.})$, exprimée au moyen des variables non accentuées, aurait tous les coefficients de x, y, z, etc. nuls. On aurait donc *toujours* $F(x', \text{etc.}) = 0$, et par suite aussi $F(x, \text{etc.}) = 0$, ce qui est contraire à l'hypothèse.

Donc enfin, on a toujours la relation (2).

5. Lemme. *Un déterminant à éléments quelconques n'a d'autre divisenr algébrique que lui même.* Théorême. *Dans la relation* (2) Δ *est une puissance du module de la transformation* (Démonstration de JORDAN, *Cours d'analyse*, pages 357–359, du tome I, ou d'ELLIOTT, dans le *Messenger* XVI, pp 7–8.

6. Application. *Le discriminant d'une fonction algébrique entiére d'autant de variables que l'on peut, l'éliminant et un système de plusieurs de ces fonctions sont des invariants* Les démonstrations habituelles, telles quelles sont données dans SALMON, *Higher Algebra*, deviennent rigoureuses, une fois qu'on les appuie sur les théorèmes précédents.

COMPARISON OF THE WEIERSTRASSIAN AND JACOBIAN ELLIPTIC FUNCTIONS.

By Prof. *Cayley.*

THE Weierstrassian function $\sigma(u)$ corresponds of course with Jacobi's $H(u)$, but it is worth while to establish the actual formulæ of transformation.

Writing, for a moment,

$$\omega = \omega_1 + iv_1,$$
$$\omega' = \omega_2 + iv_2,$$

it is convenient to assume $\omega_1 v_2 - \omega_2 v_1$ positive; we then have

$$2(\eta\omega' - \eta'\omega) = +\pi i;$$

(in particular, this will be the case if $\omega = \omega_1$, $\omega' = iv_2$ where ω_1, v_2 are each positive).

To reduce the periods into the Jacobian form, we may assume

$$\omega = \lambda K,$$
$$\omega' = \lambda i K',$$

(where observe that, if as usual k, K, K' are each real and positive, and if as above $\omega = \omega_1$, $\omega' = iv_2$, ω_1 and v_2 positive, then also λ will be real and positive). We have

$$\frac{iK'}{K} = \frac{\omega'}{\omega}, \text{ or say } q = e^{-\frac{\pi K'}{K}} = e^{\frac{2\pi\omega'}{\omega}},$$

which determines first q, and thence k, K, K', as functions of $\frac{\omega'}{\omega}$, and we then have $\lambda = \frac{\omega}{K} = \frac{-2\omega'}{K'}$, either of which equations gives λ as a function of ω, ω'. Conversely, starting with k, λ, the original equations give the value of ω, ω'; those of η, η' will be determined presently.

The form of relation is at once seen to be

$$H(u) = Ae^{Bu^2}\sigma(\lambda u),$$

and observing that for u small we have $H(u) = \sqrt{\left(\frac{2kk'K}{\pi}\right)}\,u$ and $\sigma(\lambda u) = \lambda u$, we have $A = \frac{1}{\lambda}\sqrt{\left(\frac{2kk'K}{\pi}\right)}$; I first write down and afterwards verify the value of B, viz. this is $= -\frac{1}{2}\frac{\lambda^2\eta}{\omega}$, and the formula thus is

$$H(u) = \frac{1}{\lambda}\sqrt{\left(\frac{2kk'K}{\pi}\right)}\,e^{-\frac{1}{2}\frac{\lambda^2\eta}{\omega}u^2}\sigma(\lambda u).$$

In fact, for u writing successively $u + 2K$, and $u + 2iK'$, we obtain

$$\frac{H(u+2K)}{H(u)} = e^{-\frac{\lambda^2\eta}{\omega}2K(u+K)}\;\frac{\sigma(\lambda u + 2\omega)}{\sigma(\lambda u)},$$

$$\frac{H(u+2iK')}{H(u)} = e^{-\frac{\lambda^2\eta}{\omega}2iK'(u+iK')}\frac{\sigma(\lambda u + 2\omega')}{\sigma(\lambda u)},$$

which should be satisfied in virtue of

$$\frac{H(u+2K)}{H(u)} = -1, \qquad \frac{\sigma(\lambda u + 2\omega)}{\sigma(\lambda u)} = -e^{2\eta(\lambda u + \omega)},$$

$$\frac{H(u+2iK')}{H(u)} = -e^{-\frac{i\pi}{K}(u+iK)}, \qquad \frac{\sigma(\lambda u + 2\omega')}{\sigma(\lambda u)} = -e^{2\eta'(\lambda u + \omega')},$$

viz. we ought to have

$$0=-\frac{\lambda^2\eta}{\omega}2K\,(u+K)+2\eta\,(\lambda u+\omega)$$

$$-\frac{i\pi}{K}(u+iK')=-\frac{\lambda^2\eta}{\omega}2iK'(u+iK)+2\eta'(\lambda u+\omega').$$

The first of these is

$$0=-\frac{\lambda K}{\omega}(u+K)+\left(4+\frac{\omega}{\lambda}\right),$$

that is $$0=(-1+1)(u+K);$$

and the second is

$$0=\left(\frac{\frac{1}{2}iK}{\lambda K}-\frac{\lambda iK'\eta}{\omega}\right)(u+iK')+\eta'\left(u+\frac{\omega'}{\lambda}\right),$$

viz. for $i\pi$ writing $2(\eta\omega'-\eta'\omega)$, this is

$$0=\left(\frac{\eta\omega'-\eta'\omega}{\omega}-\frac{\eta\omega'}{\omega}+\eta'\right)(u+iK'),$$

and the two equations are thus each of them an identity.

The Weierstrassian function $P(u)$ is defined as

$$=-\frac{d^2}{du^2}\log\sigma(u);$$

or, what is the same thing, we have

$$P(u)=-\frac{d}{du}\frac{\sigma'(u)}{\sigma(u)}.$$

Hence $$P(\lambda u)=-\frac{1}{\lambda}\cdot\frac{d}{du}\cdot\frac{\sigma'(\lambda u)}{\sigma(\lambda u)}.$$

But $$\frac{H'(u)}{H(u)}=-\frac{\lambda^2\eta u}{\omega}+\frac{\lambda\sigma'(\lambda u)}{\sigma(\lambda u)},$$

or $$P(\lambda u)=-\frac{\eta}{\omega}-\frac{1}{\lambda^2}\frac{d}{du}\frac{H'(u)}{H(u)}.$$

But from the equation $\sqrt{k}\,\mathrm{sn}\,u = \dfrac{H(u)}{\Theta(u)}$, we obtain

$$\frac{d}{du}\,\frac{H'(u)}{H(u)} = Z'(u) - \frac{1}{\mathrm{sn}^2 u} + k^2\,\mathrm{sn}^2 u, \quad = 1 - \frac{E}{K} - \frac{1}{\mathrm{sn}^2 u},$$

and consequently

$$P(\lambda u) = -\frac{\eta}{\omega} - \frac{1}{\lambda^2}\left(1 - \frac{E}{K}\right) + \frac{1}{\lambda^2\,\mathrm{sn}^2 u},$$

where, on the right-hand side, expanding in ascending powers of u, the constant term is

$$= -\frac{\eta}{\omega} - \frac{1}{\lambda^2}\left(1 - \frac{E}{K}\right) + \frac{1}{\lambda^2}\,\tfrac{1}{3}\,(1 + k^2).$$

But in the function $P(\lambda u)$ this constant term is $= 0$, we thus have

$$\frac{\eta}{\omega} = \frac{1}{\lambda^2}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\};$$

and then since

$$\frac{\eta}{\omega} - \frac{\eta'}{\omega'} = \frac{\frac{1}{2}\pi i}{\omega\omega'} = \frac{\frac{1}{2}\pi}{\lambda^2 KK'},$$

we have
$$\frac{\eta'}{\omega'} = -\frac{\frac{1}{2}\pi}{\lambda^2 KK'} + \frac{1}{\lambda^2}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\};$$

or, as these equations may also be written,

$$\eta = \frac{K}{\lambda}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\},$$

$$\eta' = \frac{-\frac{1}{2}\pi i}{\lambda K} + \frac{2K'}{\lambda}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\},$$

which are the values of η, η'. And we then have

$$P(\lambda u) = -\frac{1}{\lambda^2}\,\tfrac{1}{3}\,(1 + k^2) + \frac{1}{\lambda^2\,\mathrm{sn}^2 u},$$

the equation connecting $P(\lambda u)$ and $\mathrm{sn}\,u$.

ON THE DEDUCTION OF THE q-SERIES FOR THE ELLIPTIC FUNCTIONS FROM THE q-PRODUCTS.

By *J. W. L. Glaisher.*

In any comprehensive treatment of Elliptic Functions it is likely that the consideration of the q-products will precede that of the q-series, and that the latter will be deduced from the former. In a paper in vol. XVII of the *Quarterly Journal** after referring to the modes in which this transition was effected by Jacobi and Durége, I have given a systematic method of deriving the system of twelve series from the products, the main principle employed being that of resolving the products into binomial factors and performing the transformations upon these factors. This method is direct and straightforward, and for some years I have made use of it in my Lectures: but I have felt the need of a shorter and more rapid mode of obtaining the series.† The following sections contain the process by which on the whole it seems to me most convenient (at all events in the lecture room) to pass from the products to the series.

Deduction of the q-series for sn u, cn u, dn u *from the q-products*, §§1–8.

§1. Let ρ and u be used to denote respectively the two quantities $\frac{2K}{\pi}$ and $\frac{2Kx}{\pi}$. Thus we have

$$u=\frac{2Kx}{\pi}=\rho x,$$

and

$$x=\frac{\pi u}{2K}=\frac{u}{\rho}.$$

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* ‘A Chapter on Elliptic Functions,’ pp. 61–65.

† A very short method of deducing the series for sn u from the products is given in §7 of the paper referred to above, but it is somewhat artificial.

All the equations in which both u and x occur may be regarded as expressed indifferently in terms of x only, or of u only, by imagining in the respective cases that u is merely a symbol used for brevity to denote ρx, or that x is merely a symbol used to denote $\frac{u}{\rho}$.

§ 2. The given q-products are:

$$\operatorname{sn} u = \frac{1}{k^{\frac{1}{2}}}\, 2q^{\frac{1}{4}} \sin x\, \Pi_1^\infty \frac{1 - 2q^{2n} \cos 2x + q^{4n}}{1 - 2q^{2n-1} \cos 2x + q^{4n-2}},$$

$$\operatorname{cn} u = \frac{k'^{\frac{1}{2}}}{k^{\frac{1}{2}}}\, 2q^{\frac{1}{4}} \cos x\, \Pi_1^\infty \frac{1 + 2q^{2n} \cos 2x + q^{4n}}{1 - 2q^{2n-1} \cos 2x + q^{4n-2}},$$

$$\operatorname{dn} u = k'^{\frac{1}{2}}\, \Pi_1^\infty \frac{1 + 2q^{2n-1} \cos 2x + q^{4n-2}}{1 - 2q^{2n-1} \cos 2x + q^{4n-2}}.$$

§ 3. Taking the logarithms of these products, expanding the trinomial factors by means of the formula

$$\log(1 - 2a \cos 2x + a^2) = -2\Sigma_1^\infty \frac{a^n}{n} \cos 2nx,$$

and summing the series of terms which form the coefficients of $\cos 2nx$, we find

$$\log \operatorname{sn} u = \log\left(\frac{2q^{\frac{1}{4}} \sin x}{k^{\frac{1}{2}}}\right) + \Sigma_1^\infty \frac{1}{n} \frac{2q^n}{1 + q^n} \cos 2nx,$$

$$\log \operatorname{cn} u = \log\left(\frac{2k'^{\frac{1}{2}} q^{\frac{1}{4}} \cos x}{k^{\frac{1}{2}}}\right) + \Sigma_1^\infty \frac{1}{n} \frac{2q^n}{1 + (-q)^n} \cos 2nx,$$

$$\log \operatorname{dn} u = \log(k'^{\frac{1}{2}}) + \Sigma_1^\infty \frac{1}{2n-1} \frac{4q^{2n-1}}{1 - q^{4n-2}} \cos(4n-2)x.$$

§ 4. Differentiating these equations, since $du = \rho dx$, we have

$$\rho \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} = \frac{\cos x}{\sin x} - \Sigma_1^\infty \frac{4q^n}{1 + q^n} \sin 2nx,$$

$$\rho \frac{\operatorname{dn} u \operatorname{sn} u}{\operatorname{cn} u} = \frac{\sin x}{\cos x} + \Sigma_1^\infty \frac{4q^n}{1 + (-q)^n} \sin 2nx,$$

$$h\rho \frac{\operatorname{sn} u \operatorname{cn} u^{*}}{\operatorname{dn} u} = \Sigma_1^\infty \frac{8q^{2n-1}}{1 - q^{4n-2}} \sin(4n-2)x.$$

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* It is convenient to use h and h' to denote k^2 and k'^2.

§ 5. It is easily proved that

$$2\operatorname{ns}2u = \frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u} + \frac{\operatorname{dn}u\operatorname{sn}u}{\operatorname{cn}u} + h\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u},$$

$$2\operatorname{ds}2u = \frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u} + \frac{\operatorname{dn}u\operatorname{sn}u}{\operatorname{cn}u} - h\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u},$$

$$2\operatorname{cs}2u = \frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u} - \frac{\operatorname{dn}u\operatorname{sn}u}{\operatorname{cn}u} + h\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u},$$

and, by means of these formulæ, we may obtain the q-series for $\rho\operatorname{ns}2u$, $\rho\operatorname{ds}2u$, $\rho\operatorname{cs}2u$ by simply combining by addition and subtraction the q-series given in the last section.

Replacing x by $\frac{1}{2}x$ in the formulæ so obtained, we find

$$\rho\operatorname{ns}u = \frac{1}{\sin x} + \Sigma_1^\infty\frac{4q^{2n-1}}{1-q^{2n-1}}\sin(2n-1)x,$$

$$\rho\operatorname{ds}u = \frac{1}{\sin x} - \Sigma_1^\infty\frac{4q^{2n-1}}{1+q^{2n-1}}\sin(2n-1)x,$$

$$\rho\operatorname{cs}u = \frac{\cos x}{\sin x} - \Sigma_1^\infty\frac{4q^{2n}}{1+q^{2n}}\sin 2nx.$$

§ 6. Since

$$\operatorname{ns}(u+iK') = k\operatorname{sn}u,$$

$$\operatorname{ds}(u+iK') = -ik\operatorname{cn}u,$$

$$\operatorname{cs}(u+iK') = -i\operatorname{dn}u,$$

it is evident that, in order to derive the q-series for $k\rho\operatorname{sn}u$ $k\rho\operatorname{cn}u$, $\rho\operatorname{dn}u$ from those for $\rho\operatorname{ns}u$, $\rho\operatorname{ds}u$, $\rho\operatorname{cs}u$, it is only necessary to make in these formulæ the changes corresponding to the substitution of $u+iK'$ for u.

Now by the change of u into $u+iK'$, the quantity x is converted in $x+\dfrac{iK'}{\rho}$, that is into $x+\dfrac{i\pi K'}{2K}$, and therefore e^{ix} is converted into $q^{\frac{1}{2}}e^{ix}$.

§ 7. We have

$$\frac{1}{\sin x} = \frac{2i}{e^{ix}-e^{-ix}} = -2i\frac{e^{ix}}{1-e^{2ix}} = -2i\,\Sigma_1^\infty e^{(2n-1)ix},$$

and $$\frac{4q^{2n-1}}{1-q^{2n-1}}\sin(2n-1)x = -2i\frac{q^{2n-1}\{e^{(2n-1)ix}-e^{-(2n-1)ix}\}}{1-q^{2n-1}},$$

whence, by addition,

$$\rho\,\mathrm{ns}\,u = -2i\,\Sigma_1^\infty \left(e^{(2n-1)ix} + \frac{q^{2n-1}\left\{e^{(2n-1)ix} - e^{-(2n-1)ix}\right\}}{1-q^{2n-1}} \right)$$

$$= -2i\,\Sigma_1^\infty \frac{e^{(2n-1)ix} - q^{2n-1}e^{-(2n-1)ix}}{1-q^{2n-1}}\,.$$

Treating in a similar manner the q-series for $\rho\,\mathrm{ds}\,u$ and $\rho\,\mathrm{cs}\,u$, we obtain the three formulæ:

$$\rho\,\mathrm{ns}\,u = -2i\,\Sigma_1^\infty \frac{e^{(2n-1)ix} - q^{2n-1}e^{-(2n-1)ix}}{1-q^{2n-1}}\,,$$

$$\rho\,\mathrm{ds}\,u = -2i\,\Sigma_1^\infty \frac{e^{(2n-1)ix} + q^{2n-1}e^{-(2n-1)ix}}{1+q^{2n-1}}\,,$$

$$\rho\,\mathrm{cs}\,u = -i\left\{1 + 2\,\Sigma_1^\infty \frac{e^{2nix} + q^{2n}e^{-2nix}}{1+q^{2n}}\right\}.$$

§ 8. Replacing u by $u+iK'$ and e^{-ix} by $q^{\frac{1}{2}}e^{ix}$, these formulæ become

$$k\rho\,\mathrm{sn}\,u = -2i\,\Sigma_1^\infty \frac{q^{\frac{1}{2}(2n-1)}e^{(2n-1)ix} - q^{\frac{1}{2}(2n-1)}e^{-(2n-1)ix}}{1-q^{2n-1}}\,,$$

$$-ik\rho\,\mathrm{cn}\,u = -2i\,\Sigma_1^\infty \frac{q^{\frac{1}{2}(2n-1)}e^{(2n-1)ix} + q^{\frac{1}{2}(2n-1)}e^{-(2n-1)ix}}{1+q^{2n-1}}\,,$$

$$-\ i\rho\,\mathrm{dn}\,u = -\ i\left\{1 + 2\Sigma_1^\infty \frac{q^n e^{2nix} + q^n e^{-2nix}}{1+q^{2n}}\right\};$$

that is, replacing the exponentials by sines and cosines,

$$k\rho\,\mathrm{sn}\,u = \quad \Sigma_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}}\sin(2n-1)\,x,$$

$$k\rho\,\mathrm{cn}\,u = \quad \Sigma_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}}\cos(2n-1)\,x,$$

$$\rho\,\mathrm{dn}\,u = 1 + 4\Sigma_1^\infty \frac{4q^n}{1+q^{2n}}\cos 2nx,$$

which are the q-series required.

Deduction of the twelve q-series from the q-products, §§ 9, 10.

§ 9. If it be required to deduce from the q-products in § 1 not only the q-series for $k\rho \operatorname{sn} u$, $k\rho \operatorname{cn} u$, $\rho \operatorname{dn} u$, but also those for the other nine elliptic functions, the investigation is not appreciably lengthened; for the other six q-series may be deduced from the six found in §§ 5 and 8 by the simple substitution of $x + \frac{1}{2}\pi$ for x. For by this change u is converted into $u + K$, and we have

$$\begin{aligned} \operatorname{ns}(u+K) &= \operatorname{dc} u, \\ \operatorname{ds}(u+K) &= k' \operatorname{nc} u, \\ \operatorname{cs}(u+K) &= -k' \operatorname{sc} u, \end{aligned}$$

$$\begin{aligned} \operatorname{sn}(u+K) &= \operatorname{cd} u, \\ \operatorname{cn}(u+K) &= -k' \operatorname{sd} u, \\ \operatorname{dn}(u+K) &= k' \operatorname{nd} u. \end{aligned}$$

§ 10. The system of twelve q-series may be written as follows:

$$k\rho \operatorname{sn} u = \Sigma_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}} \sin(2n-1)x,$$

$$k\rho \operatorname{cn} u = \Sigma_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}} \cos(2n-1)x,$$

$$\rho \operatorname{dn} u = 1 + \Sigma_1^\infty \frac{4q^n}{1+q^{2n}} \cos 2nx;$$

$$\rho \operatorname{ns} u = \frac{1}{\sin x} + \Sigma_1^\infty \frac{4q^{2n-1}}{1-q^{2n-1}} \sin(2n-1)x,$$

$$\rho \operatorname{ds} u = \frac{1}{\sin x} - \Sigma_1^\infty \frac{4q^{2n-1}}{1+q^{2n-1}} \sin(2n-1)x,$$

$$\rho \operatorname{cs} u = \frac{\cos x}{\sin x} - \Sigma_1^\infty \frac{4q^{2n}}{1+q^{2n}} \sin 2nx;$$

$$\rho\,\mathrm{dc}\,u=\frac{1}{\cos x}+\Sigma_1^\infty\,(-)^{n-1}\frac{4q^{2n-1}}{1-q^{2n-1}}\cos(2n-1)\,x,$$

$$k'\rho\,\mathrm{nc}\,u=\frac{1}{\cos x}-\Sigma_1^\infty\,(-)^{n-1}\frac{4q^{2n-1}}{1+q^{2n-1}}\cos(2n-1)\,x,$$

$$k'\rho\,\mathrm{sc}\,u=\frac{\sin x}{\cos x}-\Sigma_1^\infty\,(-)^{n-1}\frac{4q^{2n}}{1+q^{2n}}\sin 2nx\,;$$

$$k\rho\,\mathrm{cd}\,u=\qquad\Sigma_1^\infty\,(-)^{n-1}\frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}}\cos(2n-1)\,x,$$

$$kk'\rho\,\mathrm{sd}\,u=\qquad\Sigma_1^\infty\,(-)^{n-1}\frac{4q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}}\sin(2n-1)\,x,$$

$$k'\rho\,\mathrm{nd}\,u=\quad 1-\Sigma_1^\infty\,(-)^{n-1}\frac{4q^{n}}{1+q^{2n}}\cos 2nx.$$

The twelve q-products, §§ 11, 12.

§ 11. It is obvious that the three q-products in § 1 are equivalent to the complete system of q-products for the twelve elliptic functions, for the others may be derived at sight by simple division from those given.

It seems worth while, therefore, to notice the complete system of q-series, corresponding to those in § 4, which are derivable by logarithmic differentiation from the twelve products. These are found to be:

$$\rho\,\frac{\mathrm{cs}\,u\,\mathrm{ds}\,u}{\mathrm{ns}\,u}=\rho\,\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u}=\frac{\cos x}{\sin x}-\Sigma_1^\infty\,\frac{4q^n}{1+q^n}\sin 2nx,$$

$$h'\rho\,\frac{\mathrm{sd}\,u\,\mathrm{nd}\,u}{\mathrm{cd}\,u}=h'\rho\,\frac{\mathrm{nc}\,u\,\mathrm{sc}\,u}{\mathrm{dc}\,u}=\frac{\sin x}{\cos x}-\Sigma_1^\infty(-)^{n-1}\,\frac{4q^n}{1+q^n}\sin 2nx,$$

$$\rho\,\frac{\mathrm{dn}\,u\,\mathrm{sn}\,u}{\mathrm{cn}\,u}=\rho\,\frac{\mathrm{sc}\,u\,\mathrm{dc}\,u}{\mathrm{nc}\,u}=\frac{\sin x}{\cos x}+\Sigma_1^\infty\,\frac{4q^n}{1+(-q)^n}\sin 2nx\,;$$

$$\rho\,\frac{\mathrm{ns}\,u\,\mathrm{cs}\,u}{\mathrm{ds}\,u}=\rho\,\frac{\mathrm{cd}\,u\,\mathrm{nd}\,u}{\mathrm{sd}\,u}=\frac{\cos x}{\sin x}+\Sigma_1^\infty(-)^{n-1}\,\frac{4q^n}{1+(-q)^n}\sin 2nx,$$

$$h\rho\,\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u}=h\rho\,\frac{\mathrm{sd}\,u\,\mathrm{cd}\,u}{\mathrm{nd}\,u}=\qquad\Sigma_1^\infty\,\frac{8q^{2n-1}}{1-q^{4n-2}}\sin(4n-2)\,x,$$

$$\rho\,\frac{\mathrm{ds}\,u\,\mathrm{ns}\,u}{\mathrm{cs}\,u}=\qquad\frac{\mathrm{dc}\,u\,\mathrm{nc}\,u}{\mathrm{sc}\,u}=\frac{2}{\sin 2x}+\Sigma_1^\infty\,\frac{8q^{4n-2}}{1-q^{4n-2}}\sin(4n-2)\,x.$$

§12. The complete system of formulæ corresponding to those given in §5 is:

$$2\,\mathrm{ns}\,2u = \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u} + \frac{\mathrm{dn}\,u\,\mathrm{sn}\,u}{\mathrm{cn}\,u} + h\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u},$$

$$2\,\mathrm{ds}\,2u = \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u} + \frac{\mathrm{dn}\,u\,\mathrm{sn}\,u}{\mathrm{cn}\,u} - h\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u},$$

$$2\,\mathrm{cs}\,2u = \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u} - \frac{\mathrm{dn}\,u\,\mathrm{sn}\,u}{\mathrm{cn}\,u} + h\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u};$$

$$2\,\mathrm{ns}\,2u = -\frac{\mathrm{ds}\,u\,\mathrm{cs}\,u}{\mathrm{ns}\,u} + \frac{\mathrm{cs}\,u\,\mathrm{ns}\,u}{\mathrm{ds}\,u} + \frac{\mathrm{ns}\,u\,\mathrm{ds}\,u}{\mathrm{cs}\,u},$$

$$2\,\mathrm{ds}\,2u = \frac{\mathrm{ds}\,u\,\mathrm{cs}\,u}{\mathrm{ns}\,u} - \frac{\mathrm{cs}\,u\,\mathrm{ns}\,u}{\mathrm{ds}\,u} + \frac{\mathrm{ns}\,u\,\mathrm{ds}\,u}{\mathrm{cs}\,u},$$

$$2\,\mathrm{cs}\,2u = \frac{\mathrm{ds}\,u\,\mathrm{cs}\,u}{\mathrm{ns}\,u} + \frac{\mathrm{cs}\,u\,\mathrm{ns}\,u}{\mathrm{ds}\,u} - \frac{\mathrm{ns}\,u\,\mathrm{ds}\,u}{\mathrm{cs}\,u};$$

$$2\,\mathrm{ns}\,2u = -h'\frac{\mathrm{nc}\,u\,\mathrm{sc}\,u}{\mathrm{dc}\,u} + \frac{\mathrm{sc}\,u\,\mathrm{dc}\,u}{\mathrm{nc}\,u} + \frac{\mathrm{dc}\,u\,\mathrm{nc}\,u}{\mathrm{sc}\,u},$$

$$2\,\mathrm{ds}\,2u = h'\frac{\mathrm{nc}\,u\,\mathrm{sc}\,u}{\mathrm{dc}\,u} - \frac{\mathrm{sc}\,u\,\mathrm{dc}\,u}{\mathrm{nc}\,u} + \frac{\mathrm{dc}\,u\,\mathrm{nc}\,u}{\mathrm{sc}\,u},$$

$$2\,\mathrm{cs}\,2u = -h'\frac{\mathrm{nc}\,u\,\mathrm{sc}\,u}{\mathrm{dc}\,u} - \frac{\mathrm{sc}\,u\,\mathrm{dc}\,u}{\mathrm{nc}\,u} + \frac{\mathrm{dc}\,u\,\mathrm{nc}\,u}{\mathrm{sc}\,u};$$

$$2\,\mathrm{ns}\,2u = h'\frac{\mathrm{sd}\,u\,\mathrm{nd}\,u}{\mathrm{cd}\,u} + \frac{\mathrm{nd}\,u\,\mathrm{cd}\,u}{\mathrm{sd}\,u} + h\frac{\mathrm{cd}\,u\,\mathrm{sd}\,u}{\mathrm{nd}\,u},$$

$$2\,\mathrm{ds}\,2u = h'\frac{\mathrm{sd}\,u\,\mathrm{nd}\,u}{\mathrm{cd}\,u} + \frac{\mathrm{nd}\,u\,\mathrm{cd}\,u}{\mathrm{sd}\,u} - h\frac{\mathrm{cd}\,u\,\mathrm{sd}\,u}{\mathrm{nd}\,u},$$

$$2\,\mathrm{cs}\,2u = -h'\frac{\mathrm{sd}\,u\,\mathrm{nd}\,u}{\mathrm{cd}\,u} + \frac{\mathrm{nd}\,u\,\mathrm{cd}\,u}{\mathrm{sd}\,u} - h\frac{\mathrm{cd}\,u\,\mathrm{sd}\,u}{\mathrm{nd}\,u}.$$

These groups of formulæ are deducible at sight from those given on pp. 78, 79 of the present volume.

ON CONICS SATISFYING GIVEN CONDITIONS AND TOUCHING A GIVEN CONIC.

By *R. Lachlan, B.A.*, Fellow of Trinity College, Cambridge.

In the present paper it is proposed to discuss the number of solutions which exist for each of the problems:—

To describe a conic, (i) passing through four given points and touching a given conic; (ii) passing through three given points and osculating a given conic; (iii) passing through three given points and touching a given conic at two points; (iv) passing through two given points and having contact of the fourth order with a given conic. The method employed was suggested by Clifford's paper, 'On Mr. Spottiswoode's Contact Problems,' *Philosophical Trans.*, vol. CLXIV, and Clifford's *Mathematical Papers*, pp. 287–304.; and though some of the results may be more easily deduced, the method is interesting as suggesting how similar problems may be treated when the given conic is replaced by any curve.

1. Let the tangential equations of any three points A, B, Q be

$$A \equiv x_1\lambda + y_1\mu + z_1\nu = 0,$$

$$B \equiv x_2\lambda + y_2\mu + z_2\nu = 0,$$

$$Q \equiv x_q\lambda + y_q\mu + z_q\nu = 0;$$

then the equation of any point P, lying on the conic

$$Q^2 - 4AB = 0 \text{ .... } (1),$$

will be of the form

$$P \equiv A + Q\theta + B^2\theta.$$

Hence the coordinates (x, y, z) of any point on (1) may be expressed thus

$$x = x_1 + \theta x_q + \theta^2 x_2.$$

To find the points in which (1) cuts a given conic

$$\phi(x, y, z) \equiv ax^2 + by^2 + \ldots = 0,$$

we substitute for x, y, z in terms of θ, and obtain the quartic equation

$$\phi_{2,2}\theta^4 + 2\phi_{2,q}\theta^3 + (\phi_{q,q} + 2\phi_{1,2})\,\theta^2 + 2\phi_{1,q}\theta + \phi_{1,1} = 0 \text{ .... } (2),$$

where

$$\phi_{q,r} \equiv ax_q x_r + by_q y^r + \ldots .$$

2. Writing equation (2) in the form

$$\phi_0\theta^4 + 4a_1\theta^3 + 6a_2\theta^2 + 4a_3\theta + a_4 = 0 \text{ ....} (3),$$

and denoting the symmetric functions of the differences of the roots by the usual letters we have

$$
\begin{aligned}
H &= a_0 a_2 - a_1^2, \\
G &= a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3, \\
G' &= a_4^2 a_1 - 3a_4 a_3 a_2 + 2a_3^3, \\
I &= a_0 a_4 - 4a_1 a_3 \quad + 3a_2^2, \\
J &= a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3, \\
\Delta &= I^3 - 27J^2;
\end{aligned}
$$

and we know that equation (3) has

(i) a pair of equal roots, if $\Delta = 0$,
(ii) three equal roots, if $I = J = 0$,
(iii) two pairs of equal roots, if $G = G' = 0$,
(iv) four equal roots, if $\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \frac{a_3}{a_4}$.

3. *To find the number of conics which pass through four given points and touch a given conic.*

Let the four points be A, B, C, D (fig. 10), Q the pole of AB must lie on the straight line EG; hence we may put

$$Q = E + \frac{\alpha}{\theta} G,$$

where α is indeterminate; and if the coordinates of E, G be respectively $(x_3 y_3 z_3)$, $(x_4 y_4 z_4)$, we have at once

$$\phi_{1,q} = \phi_{1,3} + \frac{\alpha}{\theta} \phi_{1,4}, \qquad \phi_{q,q} = \phi_{3,3} + \frac{2\alpha}{\theta} \phi_{3,4} + \frac{\alpha^2}{\theta^2} \phi_{4,4};$$

and substituting in (2) we have

$$
\begin{aligned}
\phi_{2,2}\theta^4 + 2\phi_{2,3}\theta^3 + (\phi_{3,3} + 2\phi_{1,2} + 2\alpha\phi_{2,4})\,\theta^2 & \\
+ 2\,(\phi_{1,3} + \alpha\phi_{3,4})\,\theta + \alpha^2\phi_{4,4} + 2\alpha\phi_{1,4} + \phi_{1,1} &= 0.
\end{aligned}
$$

For this equation, I is of the second degree in α, and J of the third, consequently Δ is of the sixth degree; hence we infer that in general *six* conics can be drawn through the points A, B, C, D to touch a given conic ϕ.

If ϕ pass through the point E, $\phi_{4,4} = 0$, and it is easy to see that Δ will now only involve α in the fifth degree, so that five conics only can be drawn to touch ϕ.

4. *To find the number of conics which pass through three given points and osculate a given conic.*

Let A, B, C be the given points, referring to the figure, we see that we may put

$$E = C + lA, \qquad G = C + mB,$$

and therefore $$Q = C + \frac{\alpha}{\theta} A + \frac{\beta}{\theta} B,$$

where α, β are indeterminate.

Denoting the coordinates of C by (x_3, y_3, z_3), we have

$$\phi_{q,1} = \phi_{1,3} + \frac{\alpha}{\theta}\phi_{1,1} + \frac{\beta}{\theta}\phi_{1,2},$$

$$\phi_{q,q} = \phi_{3,3} + \frac{\alpha^2}{\theta^2}\phi_{1,1} + \frac{\beta^2}{\theta^2}\phi_{2,2} + \frac{2\alpha\beta}{\theta^2}\phi_{1,2} + \frac{2\alpha}{\theta}\phi_{3,1} + \frac{2\beta}{\theta}\phi_{3,2};$$

substituting in (2) and reducing to the form (3) we see that a_2, a_3 involve α, β in the first degree, and a_4 involves α, β in the second degree. Hence I is of the second, J of the third degree in α, β; but in the present case we must have $I = 0$, $J = 0$; hence we can obtain six values for α, β.

Thus *six* conics can in general be drawn passing through three given points and osculating a given conic.

5. *To find the number of conics which pass through three given points and touch a given conic at two points.*

Proceeding exactly as in § 4, we see that G is of the first degree in α, β, and G' is of the fourth degree in α, β. But we must have $G = 0$, $G' = 0$: hence we obtain four values for α, β.

Thus *four* conics can in general be drawn passing through three given points and having double contact with a given conic.

6. *To find the number of conics which pass through two given points and have contact of the fourth order with a given conic.*

Let A, B be the two fixed points, C, D any other fixed points, then we can express Q thus,

$$Q = D + \frac{\alpha}{\theta} A + \frac{\beta}{\theta} B + \frac{\gamma}{\theta} C,$$

where α, β, γ are indeterminate, and proceeding as in § 4, we can show that, when equation (2) is reduced to the form (3), a_2, a_3 are of the first degree in α, β, γ, and a_4 of the second degree in α, β, γ.

But by § 3, we have

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \frac{a_3}{a_4};$$

hence to determine α, β, γ we have three equations of the degrees 1, 1, 2 respectively; hence we obtain two sets of values for α, β, γ.

Thus *two* conics can in general be drawn passing through two given conics, and having contact of the fourth order with a given conic.

7. Reciprocating these theorems we see that:—

Six conics can be drawn to touch four given straight lines and a given conic.

Six conics can be drawn to touch three given straight lines and osculating a given conic.

Four conics can be drawn to touch three given straight lines and a given conic twice.

Two conics can be drawn to touch two given straight lines and have contact of the fourth order with a given conic.

EXTENSION OF AN INVERSION PROPERTY.

By *H. M. Taylor, M.A.*

The property which it is the object of this note to generalise is the well-known one, that two curves intersect at the same angle as their inverse curves.

The generalisation may be stated as follows:—If two curves be inverted with respect to any number of centres of inversion and any radii of inversion, the curves which are the loci of the means of the corresponding points of the inverse curves intersect at the same angle as the original curves.

Let $P_1 \dots P_n$ be a series of points, whose coordinates are $a_1, b_1; \dots a_n, b_n$, and let x_1, y_1 be the coordinates of M the mean of the points $P_1 \dots P_n$, so that $nx_1 = \Sigma a$, $ny_1 = \Sigma b$; then if there be drawn lines $P_1Q_1, \dots, P_nQ_n$ of lengths $r_1, \dots, r_n$, making angles $\theta + \alpha_1 \dots \theta + \alpha_n$ with the axis of x, and x, y be the coordinates of N the mean of the points $Q_1 \dots Q_n$,

$$nx = \Sigma a + \Sigma r \cos(\theta + \alpha),$$

$$ny = \Sigma b + \Sigma r \sin(\theta + \alpha).$$

Hence, if MN make an angle ϕ with the axis of x,

$$\tan\phi = \frac{y - y_1}{x - x_1} = \frac{\Sigma r \sin(\theta + \alpha)}{\Sigma r \cos(\theta + a)}$$

$$= \frac{\sin\theta\, \Sigma r \cos\alpha + \cos\theta\, \Sigma r \sin\alpha}{\cos\theta\, \Sigma r \cos\alpha - \sin\theta\, \Sigma r \sin\alpha} = \tan(\theta + \zeta),$$

where
$$\tan\zeta = \frac{\Sigma r \sin\alpha}{\Sigma r \cos\alpha}.$$

From this we see that
$$\phi = \theta + \zeta,$$
or, in other words, that the line MN makes the same angle θ with a fixed straight line, which we may call ζ, that the lines $P_1Q_1 \ldots P_nQ_n$ make with fixed straight lines, which we may call $\alpha_1 \ldots \alpha_n$ respectively.

It will be noticed that the angle θ must be measured in the same direction from each of the fixed straight lines.

Now if PQ be an element of a curve and P_mQ_m the element of the inverse curve with respect to the centre of inversion O_m, the lines PQ, P_mQ_m make the same angle with OPP_m on opposite sides of it, and
$$\frac{P_mQ_m}{O_mP_m} = \frac{PQ}{O_mP}, \text{ or } P_mQ_m = \frac{O_mP_m}{O_mP} PQ.$$

It will easily be seen that, if $P_1Q_1, \ldots, P_nQ_n$ be the inverse elements of PQ with respect to the series of centres of inversion $O_1, \ldots O_n$, then $P_1Q_1, \ldots, P_nQ_n$, and therefore MN, all make the same angle, measured in the same direction, with fixed straight lines that PQ makes with a certain fixed straight line, measured in the opposite direction. Hence, if PQ, PQ' be elements of two curves passing through P and MN, MN' the corresponding elements passing through M, the angles QPQ', NMN' are equal.

This proves the generalisation as stated above, but the proposition admits of still further generalisation. For it may be remarked that the inversion with respect to each centre of inversion may be repeated any number of times, or, in other words that different weights may be given to the inverted curves.

Further, it may be remarked that successive inversions may be taken; for instance the two original curves may be inverted p times successively with respect to p different centres of inversion, q times successively with respect to q different centres, r times successively with respect to r different centres, and so on; and the proposition will still be true that the curves, which are the loci of the means of the corresponding points of the curves obtained by these sets of successive inversions, intersect at the same angle as the original curves, with the one proviso that p, q, r,...must be all even or all odd.

ON THE PROCESS OF SQUARING THE q-SERIES FOR $k\rho\,\mathrm{sn}\,u$, $k\rho\,\mathrm{cn}\,u$, $\rho\,\mathrm{dn}\,u$.

By *J. W. L. Glaisher.*

The q-series for $k\rho\,\mathrm{sn}\,u$, &c., and $k^2\rho^2\,\mathrm{sn}^2u$, &c., §§ 1, 2.

§ 1. Putting $\rho = \dfrac{2K}{\pi}$, $u = \dfrac{2Kx}{\pi}$,

the q-series for $k\rho\,\mathrm{sn}\,u$, $k\rho\,\mathrm{cn}\,u$, $\rho\,\mathrm{dn}\,u$ are:

$$k\rho\,\mathrm{sn}\,u = \Sigma_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}} \sin(2n-1)x,$$

$$k\rho\,\mathrm{cn}\,u = \Sigma_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}} \cos(2n-1)x,$$

$$\rho\,\mathrm{dn}\,u = 1 + \Sigma_1^\infty \frac{4q^n}{1+q^{2n}} \cos 2nx;$$

and the q-series for the squares of these functions are

$$k^2\rho^2\,\mathrm{sn}^2u = -\frac{4KI}{\pi^2} - \Sigma_1^\infty \frac{8nq^n}{1-q^{2n}} \cos 2nx,$$

$$k^2\rho^2\,\mathrm{cn}^2u = \frac{4KG}{\pi^2} + \Sigma_1^\infty \frac{8nq^n}{1-q^{2n}} \cos 2nx,$$

$$\rho^2\,\mathrm{dn}^2u = \frac{4KE}{\pi^2} + \Sigma_1^\infty \frac{8nq^n}{1-q^{2n}} \cos 2nx,$$

where

$$\frac{4KI}{\pi^2} = -\Sigma_1^\infty \frac{8q^{2n-1}}{(1-q^{2n-1})^2} = -\Sigma_1^\infty \frac{8nq^n}{1-q^{2n}},$$

$$\frac{4KG}{\pi^2} = \Sigma_1^\infty \frac{8q^{2n-1}}{(1+q^{2n-1})^2} = \Sigma_1^\infty (-)^{n-1} \frac{8nq^n}{1-q^{2n}},$$

$$\frac{4KE}{\pi^2} = 1 + \Sigma_1^\infty \frac{8q^{2n}}{(1+q^{2n})^2} = 1 + \Sigma_1^\infty (-)^{n-1} \frac{8nq^{2n}}{1-q^{2n}},$$

and $I = E - K$, $G = E - k'^2K$.

§ 2. In § 41 of the *Fundamenta Nova* Jacobi obtained the q-series for $k^2\rho^2\,\mathrm{sn}^2 u$ by multiplying the q-series for $k\rho\,\mathrm{sn}\,u$ by itself. Jacobi's process is reproduced by Cayley in his *Elliptic Functions*, pp. 291–293.

The object of the present note is to show how the series for the squared functions may be derived from the series for the simple functions by a process of squaring which seems more simple and straightforward than that employed by Jacobi.

It is immaterial for which of the three squared functions we obtain a q-series, as the other two are deducible at once from any one of them by the formulæ

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \qquad k^2\,\mathrm{sn}^2 u + \mathrm{dn}^2 u = 1.$$

Selecting the q-series for $\rho\,\mathrm{dn}\,u$ we may obtain its square in the following manner.

Squaring the q-series for $\rho\,\mathrm{dn}\,u$, § 3.

§ 3. Let
$$\alpha = \frac{\pi K'}{K}, \quad v = e^{2ix};$$

the q-series for $\rho\,\mathrm{dn}\,u$ may then be written

$$\rho\,\mathrm{dn}\,u = \Sigma_{-\infty}^{\infty} \frac{v^n}{\cosh n\alpha}.$$

Let
$$\rho^2\,\mathrm{dn}^2 u = \Sigma_{-\infty}^{\infty}\, C_n v^n;$$

therefore
$$C_n = \Sigma_{r=-\infty}^{r=\infty} \frac{1}{\cosh r\alpha \cosh (n-r)\,\alpha}.$$

Now

$$\frac{1}{\cosh r\alpha \cosh (n-r)\,\alpha} = \frac{1}{\sinh n\alpha}\,\frac{\sinh \{r\alpha + (n-r)\,\alpha\}}{\cosh r\alpha \cosh (n-r)\,\alpha}$$
$$= \frac{1}{\sinh n\alpha}\,\{\tanh r\alpha + \tanh (n-r)\,\alpha\}$$
$$= \frac{1}{\sinh n\alpha}\,\{\tanh r\alpha - \tanh (r-n)\,\alpha\}.$$

Therefore

$$C_n = \frac{1}{\sinh n\alpha}\,\Sigma_{r=-\infty}^{r=\infty}\,\{\tanh r\alpha - \tanh (r-n)\,\alpha\}.$$

Now, denoting by t and t' any positive integers, and supposing n positive and $t+t'>n$, then

$$\Sigma_{r=-t'}^{r=t}\{\tanh r\alpha - \tanh(r-n)\alpha\}$$

$$= \tanh t\alpha + \tanh(t-1)\alpha + \ldots + \tanh(t-n+1)\alpha$$

$$+ \tanh(t'+n)\alpha + \tanh(t'+n-1)\alpha + \ldots + \tanh(t'+1)\alpha.$$

This is evident, for the arguments in the series

$$\Sigma_{r=-t'}^{r=t}\tanh r\alpha$$

are the numbers from t to $-t'$, and the arguments in the series

$$\Sigma_{r=-t'}^{r=t}\tanh(r-n)\alpha$$

are the numbers from $t-n$ to $-t'-n$. Thus the series overlap each other except as regards the first n terms of the former and the last n terms of the latter.

When t and t' are infinite each of these hyperbolic tangents is equal to unity; and since they are $2n$ in number, we have

$$C_n = \frac{2n}{\sinh n\alpha},$$

giving
$$\rho^2\,\mathrm{dn}^2 u = C_0 + \Sigma_1^\infty \frac{4n\cos 2nx}{\sinh n\alpha}$$

$$= C_0 + \Sigma_1^\infty \frac{8nq^n}{1-q^{2n}}\cos 2nx.$$

The quantity C_0, which $= \Sigma_{-\infty}^{\infty}\dfrac{1}{\cosh^2 r\alpha}$, may be expressed in terms of E by integration between the limits 0 and $\frac{1}{2}\pi$ exactly as in the *Fundamenta Nova.*

Squaring the q-series for $k\rho$ cn u, § 4.

§ 4. The q-series for $k\rho\,\mathrm{cn}\,u$ may be squared in a similar manner. Putting

$$a = \tfrac{1}{2}\frac{\pi K'}{K}, \quad z = e^{ix},$$

(so that a and z are respectively the half and the square root of α and v used in the last section); we have

$$k\rho\,\mathrm{cn}\,u = \Sigma'^{\infty}_{-\infty}\frac{z^n}{\cosh na},$$

where the accent attached to Σ denotes that n is only to have uneven values assigned to it.

Let $$k^2\rho^2\,\mathrm{cn}^2 u = \Sigma_{-\infty}^{\infty} B_n z^{2n};$$

then $$B_n = \Sigma'^{s=\infty}_{s=-\infty} \frac{1}{\cosh sa \cosh(2n-s)a}$$

$$= \frac{1}{\sinh 2na}\,\Sigma'^{s=\infty}_{s=-\infty} \{\tanh sa - \tanh(s-2n)a\}.$$

Now $$\Sigma'^{s=t}_{s=-t}\{\tanh sa - \tanh(s-2n)a\}$$

$$= \tanh ta + \tanh(t-2)a + \ldots + \tanh(t-2n+2)a$$

$$+ \tanh(t'+2n)a + \tanh(t'+2n-2)a + \ldots + \tanh(t'+2)a$$

$= 2n$, when t and t' are infinite;

and therefore $$B_n = \frac{2n}{\sinh 2na},$$

giving $$k^2\rho^2\,\mathrm{cn}^2 u = B_0 + \Sigma_1^{\infty} \frac{8nq^n}{1-q^{2n}} \cos 2nx.$$

Squaring the q-series for $k\rho\,\mathrm{sn}\,u$, § 5.

§ 5. Using the same notation as in the last section,

$$k\rho\,\mathrm{sn}\,u = \frac{1}{i}\,\Sigma'^{\infty}_{-\infty} \frac{z^n}{\sinh na}.$$

Let $$k^2\rho^2\,\mathrm{sn}^2 u = \Sigma_{-\infty}^{\infty} A_n z^{2n},$$

then $$A_n = -\,\Sigma'^{s=\infty}_{s=-\infty} \frac{1}{\sinh sa \sinh(2n-s)a}$$

$$= -\frac{1}{\sinh 2na}\,\Sigma'^{s=\infty}_{s=-\infty} \{\coth sa - \coth(s-2n)a\}$$

$$= -\frac{2n}{\sinh 2na};$$

and we find

$$k^2\rho^2\,\mathrm{sn}^2 u = A_0 - \Sigma_1^{\infty} \frac{8nq^n}{1-q^{2n}} \cos 2nx.$$

Résumé, §§ 6, 7.

§ 6. It will have been seen that the process of squaring is a little simpler in the case of the series for $\rho\,\mathrm{dn}\,u$ than in that of the series for $k\rho\,\mathrm{cn}\,u$ or $k\rho\,\mathrm{sn}\,u$. The series for $\rho\,\mathrm{dn}\,u$ is therefore the best of the three to select for squaring: the process was exhibited in § 3 in the notation which seemed the simplest, α and v being used instead of $2a$ and z^2. It may be convenient however to give a *résumé* of the process of squaring for the three functions, using the same notation for all; viz.

$$a = \tfrac{1}{2}\frac{\pi K'}{K}, \qquad z = e^{ix}.$$

$$k\rho\,\mathrm{sn}\,u = \frac{1}{i}\,\Sigma'^{\infty}_{-\infty}\,\frac{z^n}{\sinh na},$$

$$k\rho\,\mathrm{cn}\,u = \Sigma'^{\infty}_{-\infty}\,\frac{z^n}{\cosh na},$$

$$\rho\,\mathrm{dn}\,u = \Sigma^{\infty}_{-\infty}\,\frac{z^{2n}}{\cosh 2na};$$

$$k^2\rho^2\,\mathrm{sn}^2 u = \Sigma^{\infty}_{-\infty} A_n z^{2n},$$

$$k^2\rho^2\,\mathrm{cn}^2 u = \Sigma^{\infty}_{-\infty} B_n z^{2n},$$

$$\rho^2\,\mathrm{dn}^2 u = \Sigma^{\infty}_{-\infty} C_n z^{2n};$$

$$A_n = -\,\Sigma'^{s=\infty}_{s=-\infty}\,\frac{1}{\sinh sa \sinh (2n-s)\,a} = -\,\frac{2n}{\sinh 2na},$$

$$B_n = \Sigma'^{s=\infty}_{s=-\infty}\,\frac{1}{\cosh sa \cosh (2n-s)\,a} = \frac{2n}{\sinh 2na},$$

$$C_n = \Sigma^{r=\infty}_{r=-\infty}\,\frac{1}{\cosh 2ra \cosh (2n-2r)\,a} = \frac{2n}{\sinh 2na}.$$

(Σ includes all integral values, Σ' includes only uneven values.)

ON THE PROCESS OF SQUARING THE ZETA FUNCTION.

By *J. W. L. Glaisher.*

§ 1. The process by which the q-series for $\rho\,\mathrm{dn}\,u$ was squared in the preceding note applies with very slight modifications to the Zeta Function also.*

§ 2. Putting as before

$$\rho = \frac{2K}{\pi}, \qquad u = \frac{2Kx}{\pi},$$

the q-series for $\rho Z(u)$ and $\rho^2 Z^2(u)$ are respectively:

$$\rho Z(u) = \Sigma_1^\infty \frac{4q^n}{1-q^{2n}} \sin 2nx,$$

$$\rho^2 Z^2(u) = M_0 + \Sigma_1^\infty \left\{ \frac{8q^n (1+q^{2n})}{(1-q^{2n})^2} - \frac{8nq^n}{1-q^{2n}} \right\} \cos 2nx,$$

where

$$M_0 = \Sigma_1^\infty \frac{8q^{2n}}{(1-q^{2n})^2} = \Sigma_1^\infty \frac{8nq^{2n}}{1-q^{2n}}$$

$$= \tfrac{1}{3} - \frac{4K(I+G+E)}{3\pi^2}.$$

§ 3. Using the same notation as in § 3 of the preceding note, we put

$$\alpha = \frac{\pi K'}{K}, \qquad v = e^{2ix};$$

whence $$\rho Z(u) = \frac{1}{i} \Sigma_{-\infty}^{\infty} \frac{v^n}{\sinh n\alpha}$$

(the value $n=0$ being excluded).

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* Jacobi's method of squaring the q-series for the Zeta Function is contained in § 48 of the *Fundamenta Nova.*

Let
$$\rho^2 Z^2(u) = \Sigma_{-\infty}^{\infty} M_n v^n;$$

then
$$M_n = -\Sigma_{r=-\infty}^{r=\infty} \frac{1}{\sinh r\alpha \sinh (n-r)\alpha}$$
(the values $r=0$ and $r=n$ being excluded)

$$= -\frac{1}{\sinh n\alpha} \Sigma_{r=-\infty}^{r=\infty} \{\coth r\alpha - \coth (r-n)\alpha\}$$
($r=0$ and $r=n$ excluded).

Now, supposing n positive, $t > 2n$, and $t' > n$,

$\Sigma_{r=-t'}^{r=t} \{\coth r\alpha - \coth (r-n)\alpha\}$ ($r=0$ and $r=n$ excluded)

$$\begin{aligned} = \quad & \coth t\alpha + \coth (t-1)\alpha + \ldots + \coth (t-n+1)\alpha \\ & - \coth n\alpha - \coth n\alpha \\ & + \coth (t'+n)\alpha + \coth (t'+n-1)\alpha + \ldots + \coth (t'+1)\alpha. \end{aligned}$$

The terms in the second line are the values of $\coth r\alpha$ for $r=-n$ and of $-\coth (r-n)\alpha$ for $r=2n$. Since the values $r=0$ and $r=n$ are excluded, the terms which otherwise would neutralise these two terms do not occur. Thus

$$M_n = -\frac{1}{\sinh n\alpha} \{2n - 2 \coth nx\},$$

and therefore
$$\rho^2 Z^2(u) = M_0 - \Sigma_{1}^{\infty} \left\{ n - \frac{1+q^{2n}}{1-q^{2n}} \right\} \frac{8q^n}{1-q^{2n}} \cos 2nx.$$

§ 4. The process is therefore exactly analogous to that by which the square of $\rho \operatorname{dn} u$ was obtained, except that owing to the exclusion of the value $n=0$ in the original series two terms in the middle, in addition to the n terms at each end, remain uncancelled.

ON PORISTIC SYSTEMS OF CIRCLES.

By *R. Lachlan, B.A.*, Fellow of Trinity College, Cambridge.

In Gergonne's *Annales*, vol. XIX., Steiner proposes for solution the problem 'to find the condition that, given any two circles, a ring of circles may be drawn to touch the given circles, and each of its neighbours in the series'; this condition he states to be

$$(R \mp r)^2 \mp 4Rr \tan^2 \frac{n\pi}{m} = d^2 ;$$

where R, r are the radii of the two given circles, d the distance between their centres, n the number of circles in the series, m the number of times the ring encircles the given circle of smaller radius; and the upper or lower sign being taken according as one of the two given circles lies within or without the other.

This problem is discussed, and an equivalent condition deduced by H. M. Taylor (*Messenger*, vol. VII. p. 148); and some further results are given by W. W. Taylor in the same volume of the Messenger (pp. 167—170).

I propose in this paper to discuss the more general case, where each circle of the series cuts its neighbours at a constant angle, and also two given circles each at constant angles.

General Theory, §§ 1–8.

1. Let A, B be any two circles (fig. 11), B lying entirely within A. Let r, r' be their radii, d the distance between their centres, and ω their angle of intersection—this angle is imaginary, and we shall find it convenient to make use of it, but our results may be more simply expressed by taking an angle ϖ, defined by the equation

$$(r - r')^2 - d^2 = 4rr' \tan^2 \varpi ,$$

or

$$1 + \cos \omega + 2 \tan^2 \varpi = 0 \quad \text{..................(1).}$$

2. Let any other circle (1) be drawn cutting A, B at angles α, β; and let us consider a series of circles (1, 2, 3, ...), such that each circle (n) cuts A, B at the same angles α, β; and its two neighbours in the series ($n + 1$, $n - 1$) at the angle ϕ.

The problems which suggest themselves are: i. To find r_n in terms of r, r', r_1; ii. the angle ζ_n at which the circle (n) intersects any circle (z); iii. under what circumstances the system is poristic.

It will be seen that if α, β, ϕ are real, there is a poristic system in general; if α, β are each right angles, the system is poristic for all real values of ϕ, whatever be the position of A, B. But there is an exceptional case, viz. when A, B touch, in which case a poristic system is impossible.

3. Now denoting the power of two circles (p, q) by $\pi_{p,q}$, we have (see *Messenger*, vol. XVI. p. 99) the formula

$$\pi\begin{pmatrix}1, & 2, & 3, & 4, & 5\\ 6, & 7, & 8, & 9, & 10\end{pmatrix} = 0 \;\ldots\ldots\ldots\ldots\ldots\; (2).$$

In this formula any circle may be replaced by the line at infinity, which may be denoted by θ, and then we have

$$\pi_{1,2} = 2r_1r_2\cos\phi_{1,2},$$

$$\pi_{1,\theta} = 1, \qquad \pi_{\theta,\theta} = 0,$$

$\phi_{1,2}$ being the angle of intersection of the circles $(1, 2)$.

4. Let (p) denote either of the circles $(n-1, n+1)$, then by (2)

$$\pi\begin{pmatrix}\theta, & p, & n, & A, & B\\ \theta, & p, & n, & A, & B\end{pmatrix} = 0,$$

which reduces to

$$\begin{vmatrix} 0, & \frac{1}{r_p}, & \frac{1}{r_n}, & \frac{1}{r}, & \frac{1}{r'} \\ \frac{1}{r_p}, & -1, & \cos\phi, & \cos\alpha, & \cos\beta \\ \frac{1}{r_n}, & \cos\phi, & -1, & \cos\alpha, & \cos\beta \\ \frac{1}{r}, & \cos\alpha, & \cos\beta, & -1, & \cos\omega \\ \frac{1}{r'}, & \cos\alpha, & \cos\beta, & \cos\omega, & -1 \end{vmatrix} = 0.$$

This is a quadratic in r_p, the two roots being r_{n+1}, r_{n-1}.

We have at once

$$2S\left(\frac{1}{r_{n+1}}+\frac{1}{r_{n-1}}\right)$$

$$=\quad \frac{1}{r_n}(\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta\cos\omega+\cos\phi\sin^2\omega)$$

$$+\frac{1}{r}(1+\cos\phi)(\cos\alpha+\cos\beta\cos\omega)$$

$$+\frac{1}{r'}(1+\cos\phi)(\cos\beta+\cos\alpha\cos\omega)\;\ldots\ldots\ldots\ldots\;(3),$$

where

$$4S=\cos^2\alpha+\cos^2\beta+\cos^2\omega+2\cos\alpha\cos\beta\cos\omega-1.$$

Let us write

$$\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta\cos\omega+\cos\phi\sin^2\omega=4S\cos\psi,$$

so that $$4S(1-\cos\psi)+\sin^2\omega(1+\cos\phi)=0\;\ldots\ldots\ldots(4).$$

Then (3) may be written in the form

$$\frac{1}{r_{n+1}}-2\cos\psi\frac{1}{r_n}+\frac{1}{r_{n-1}}$$

$$=-2\frac{1-\cos\psi}{\sin^2\omega}\left\{\frac{1}{r}(\cos\alpha+\cos\beta\cos\omega)+\frac{1}{r'}(\cos\beta+\cos\alpha\cos\omega)\right\},$$

the solution of which is

$$\frac{1}{r_n}=M\cos n\psi+N\sin n\psi$$

$$-\frac{\cos\alpha+\cos\beta\cos\omega}{\sin^2\omega}\frac{1}{r}-\frac{\cos\beta+\cos\alpha\cos\omega}{\sin^2\omega}\frac{1}{r'}\;\ldots\ldots(5),$$

except when $\psi=0$, which can only occur when $\omega=0$ or π, *i.e.* when the circles A, B touch; but, since B is supposed to be within A, the value $\omega=0$ is inadmissible; hence $\omega=\pi$, and the solution is

$$\frac{1}{r_n}=M+Nn+\frac{1+\cos\phi}{4S}\left\{\frac{\cos\alpha-\cos\beta}{r}+\frac{\cos\beta-\cos\alpha}{r'}\right\}n^2;$$

or, since $$4S=\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta,$$

$$\frac{1}{r_n}=M+Nn+n^2\frac{1+\cos\phi}{\cos\alpha-\cos\beta}\left(\frac{1}{r}-\frac{1}{r'}\right)\;\ldots\ldots(6).$$

5. Again, from (2) we have, if z denote any circle whatever,

$$\pi\begin{pmatrix} z, & p, & n, & A, & B \\ z, & p, & n, & A, & B \end{pmatrix} = 0.$$

Let z cut A, B at angles γ, δ, and the circle (n) at the angle ζ_n, then this equation becomes

$$\begin{vmatrix} -1\,, & \cos\zeta_p, & \cos\zeta_n, & \cos\gamma, & \cos\delta \\ \cos\zeta_p, & -1\,, & \cos\phi, & \cos\alpha, & \cos\beta \\ \cos\zeta_n, & \cos\phi, & -1\,, & \cos\alpha, & \cos\beta \\ \cos\gamma\,, & \cos\alpha, & \cos\alpha, & -1\,, & \cos\omega \\ \cos\delta\,, & \cos\beta, & \cos\beta, & \cos\omega, & -1 \end{vmatrix} = 0,$$

i.e. a quadratic to find $\cos\zeta_p$, the roots of which are $\cos\zeta_{n+1}$, $\cos\zeta_{n-1}$; and we have at once, by the same process as in § 4,

$$\cos\zeta_{n-1} - 2\cos\psi\cos\zeta_n + \cos\zeta_{n-1}$$
$$= -2\,\frac{1-\cos\psi}{\sin^2\omega}\,\{(\cos\alpha + \cos\beta\cos\omega)\cos\gamma$$
$$+ (\cos\beta + \cos\alpha\cos\omega)\cos\delta\}.$$

The solution of this equation is

$$\cos\zeta_n = M'\cos n\psi + N'\sin n\psi - \operatorname{cosec}^2\omega\,\{(\cos\alpha + \cos\beta\cos\omega)\cos\gamma$$
$$+ (\cos\beta + \cos\alpha\cos\omega)\cos\delta\} \ldots (7),$$

except when $\psi = 0$, in which case we have

$$\cos\zeta_n = M' + N'n + n^2(1+\cos\phi)\,\frac{\cos\gamma - \cos\delta}{\cos\alpha - \cos\beta} \ldots\ldots (8).$$

6. To determine the constants in these equations, the simplest method is to first find r_2 and ζ_2.

To find r_2, let C denote the circle which cuts the circles A, B (1) orthogonally, and let R be its radius; also let C cut (2) at the angle χ.

From (2) we have

$$\pi\begin{pmatrix} C, & 2, & 1, & A, & B \\ C, & 2, & 1, & A, & B \end{pmatrix} = 0,$$

i.e.

$$\begin{vmatrix} \frac{1}{R}\,, & \frac{1}{r_2}\,, & \frac{1}{r_1}\,, & \frac{1}{r}\,, & \frac{1}{r} \\ \cos\chi, & -1\,, & \cos\phi, & \cos\alpha\,, & \cos\beta \\ 0\,, & \cos\phi, & -1\,, & \cos\alpha\,, & \cos\beta \\ 0\,, & \cos\alpha, & \cos\alpha\,, & -1\,, & \cos\omega \\ 0\,, & \cos\beta, & \cos\beta, & \cos\omega, & -1 \end{vmatrix} = 0,$$

or

$$\frac{1}{R}\begin{vmatrix} -1, & \cos\phi, & \cos\alpha, & \cos\beta \\ \cos\phi, & -1, & \cos\alpha, & \cos\beta \\ \cos\alpha, & \cos\alpha, & -1, & \cos\omega \\ \cos\beta, & \cos\beta, & \cos\omega, & -1 \end{vmatrix}$$

$$= \cos\chi \begin{vmatrix} \frac{1}{r_2}, & \frac{1}{r_1}, & \frac{1}{r}, & \frac{1}{r'} \\ \cos\phi, & -1, & \cos\alpha, & \cos\beta \\ \cos\alpha, & \cos\alpha, & -1, & \cos\omega \\ \cos\beta, & \cos\beta, & \cos\omega, & -1 \end{vmatrix} \text{.............(9).}$$

And again, from (2), we have

$$\pi\begin{pmatrix} C, & 2, & 1, & A, & B \\ C, & 2, & 1, & A, & B \end{pmatrix} = 0,$$

i.e

$$\begin{vmatrix} -1, & \cos\chi, & 0, & 0, & 0 \\ \cos\chi, & -1, & \cos\phi, & \cos\alpha, & \cos\beta \\ 0, & \cos\phi, & -1, & \cos\alpha, & \cos\beta \\ 0, & \cos\alpha, & \cos\alpha, & -1, & \cos\omega \\ 0, & \cos\beta, & \cos\beta, & \cos\omega, & -1 \end{vmatrix} = 0;$$

whence

$$4S\cos^2\chi + \begin{vmatrix} -1, & \cos\phi, & \cos\alpha, & \cos\beta \\ \cos\phi, & -1, & \cos\alpha, & \cos\beta \\ \cos\alpha, & \cos\alpha, & -1, & \cos\omega \\ \cos\beta, & \cos\beta, & \cos\omega, & -1 \end{vmatrix} = 0 \text{.......(10).}$$

From (9) and (10) we have

$$4S\left(\frac{1}{r_2} + \frac{\cos\chi}{R}\right)$$

$$= \frac{1}{r_1}\left(\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos\omega + \cos\phi\sin^2\alpha\right)$$

$$+ \frac{1}{r}(1+\cos\phi)(\cos\alpha + \cos\beta\cos\omega)$$

$$+ \frac{1}{r'}(1+\cos\phi)(\cos\beta + \cos\alpha\cos\omega),$$

or

$$\frac{1}{r_2}=\frac{\cos\psi}{r_1}-\frac{\cos\chi}{R}-\frac{1-\cos\psi}{\sin^2\omega}$$

$$\times\left\{\frac{1}{r}(\cos\alpha+\cos\beta\cos\omega)+\frac{1}{r'}(\cos\beta+\cos\alpha\cos\omega)\right\}\ldots(11).$$

From (10) we easily find that

$$\cos^2\chi=(1+\cos\phi)(1+\cos\psi),$$

or if we suppose (2) to be drawn on the side of (1) remote to the radical centre of (1, A, B), so that $\cos\chi$ is negative, we have

$$\cos\chi=-2\cos\tfrac{1}{2}\phi\cos\tfrac{1}{2}\psi\ldots\ldots\ldots\ldots\ldots(12).$$

We should also have, by taking

$$\pi\begin{pmatrix}C, & 2, & 1, & A, & B\\ z, & 2, & 1, & A, & B\end{pmatrix}=0,$$

the formula

$$\cos\zeta_2=\cos\psi\cos\zeta_1-\cos\chi\cos\zeta-\frac{1-\cos\psi}{\sin^2\omega}$$

$$\{(\cos\alpha+\cos\beta\cos\omega)\cos\gamma+(\cos\beta+\cos\alpha\cos\omega)\cos\delta\}\ldots(13),$$

where ζ is the angle at which the circle z intersects the circle C.

7. Reserving for the present the case where A, B touch, we find for the constants M, N in (5), the values

$$M=\left(\frac{1}{r_1}+K\right)\cos\psi-\frac{2}{R}\cos\tfrac{1}{2}\phi\cos\tfrac{1}{2}\psi,$$

$$N=\left(\frac{1}{r_1}+K\right)\sin\psi+\frac{2}{R}\cos\psi\cos\tfrac{1}{2}\phi\cos\tfrac{1}{2}\psi,$$

where

$$K\sin^2\omega=\frac{1}{r}(\cos\alpha+\cos\beta\cos\omega)+\frac{1}{r'}(\cos\beta+\cos\alpha\cos\omega).$$

Hence equation (5) may be written

$$\frac{1}{r_n}+K=\left(\frac{1}{r_1}+K\right)\cos(n-1)\psi+\frac{1}{R}\cos\tfrac{1}{2}\phi\,\frac{\sin(n-1)\psi}{\sin\tfrac{1}{2}\psi}\ldots(14).$$

8. Similarly we shall find that equation (7) may be written

$$\cos\zeta_n + K' = (\cos\zeta_1 + K')\cos(n-1)\psi$$

$$\cos\zeta\cos\tfrac{1}{2}\phi\,\frac{\sin(n-1)\psi}{\sin\frac{1}{2}\psi} \quad \text{...(15),}$$

where

$$K'\sin^2\omega = \cos\gamma(\cos\alpha + \cos\beta\cos\omega) + \cos\delta(\cos\beta + \cos\alpha\cos\omega).$$

Two particular cases may be noticed:

i. If z is the orthogonal circle of the system $(A, B, 1)$, in which case we have

$$\cos\zeta_n = -\cos\tfrac{1}{2}\phi\operatorname{cosec}\tfrac{1}{2}\psi\sin(n-1)\psi.$$

ii. If z coincides with (1), in which case

$$\cos\zeta_n = \frac{\cos\psi + \cos\phi - (1+\cos\phi)\cos(n-1)\psi}{1-\cos\psi}.$$

Poristic systems, §§ 9–13.

9. The form of equation (5) shews that, if ψ, as given by (4), is a real angle, the system of circles consists of a finite number; thus, if $\psi = \frac{2m\pi}{n}$, the system consists of n circles encircling the inner circle B m times.

Let us express ψ in terms of the real angle ϖ given in (1); we have

$$4S = \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta(1 + 2\tan^2\varpi) + 4\tan^2\varpi\sec^2\varpi;$$

and (4) may be written

$$\frac{\cos^2\frac{1}{2}\phi}{\sin^2\frac{1}{2}\psi} = \frac{(\cos\alpha - \cos\beta)^2 - 4\cos\alpha\cos\beta\tan^2\varpi + 4\tan^2\varpi\sec^2\varpi}{4\tan^2\varpi\sec^2\varpi} \quad \text{............(16).}$$

In general ψ will be a real angle, when ϕ is a real angle, though it would be possible to choose α, β, ϕ, so that ψ may be real even when ϕ is not.

10. A few special cases may be noticed

i. If $\alpha=\beta$,

$$\sin^2\tfrac{1}{2}\psi=\frac{\cos^2\frac{1}{2}\phi}{1-\cos^2\alpha\cos^2\varpi},$$

ψ is real, only when ϕ is real; and

$$\sin^2\tfrac{1}{2}\phi>\cos^2\alpha\cos^2\varpi.$$

ii. If $\alpha=\beta=\frac{1}{2}\pi$, then, whatever the value of ϖ (except $\varpi=0$, which is excluded for the present), we have

$$\cos\psi+\cos\phi=0,$$

or

$$\phi+\psi=\pi.$$

iii. If $\alpha=\pi$, $\beta=0$, we have

$$\sin^2\tfrac{1}{2}\psi=\sin^2\varpi\cos^2\tfrac{1}{2}\phi,$$

and if $\phi=0$, this gives $\varpi=\frac{1}{2}\psi$.

11. If ψ given by equation (16) be a real angle, commensurable with π, we infer that given any two circles A, B, then any circle (1) being drawn as in fig. (11), cutting them at angles α, β respectively, and also a series of circles (2, 3, ...), such that any circle (n) cuts A at the angle α, and B at the angle β, and its two neighbours in the series at the angle ϕ, the number in the series will be limited; and the number will be the same whatever the position of the first circle in the series.

Again, if we suppose α, β to be given, we can by (16) always determine the angle ϕ, so that the number of circles in the series may be equal to any number we please. For example, if we wish to find ϕ, so that the system may form a ring of n circles, encircling B m times, we have merely to put $\psi=\dfrac{2m\pi}{n}$ in (16).

Again, if we take α, β, ϕ as given, we can determine ϖ, so that the system may form a ring of n circles encircling B m times; that is, given A and B in magnitude, we can find at what distance apart their centres must be placed in order that such a system may exist.

12. In the particular case, when $\alpha=\beta=\frac{1}{2}\pi$, we found that

$$\phi+\psi=\pi.$$

Hence, given any two circles A, B placed in any position whatever (except touching each other), then if a series of circles be drawn, say (1, 2, 3, ...), each cutting A and B orthogonally, and its two neighbours in the series at the constant angle $\frac{2\pi}{n}$, the system will consist of m circles, where

$$\frac{1}{m} + \frac{1}{n} = \tfrac{1}{2}.$$

13. In the case when $\alpha = \pi$, $\beta = 0$, $\phi = 0$, we have $\varpi = \frac{1}{2}\psi$; hence, if the system of circles consists of n circles, we have

$$\varpi = \frac{m\pi}{n},$$

and, by (1), $$(r - r')^2 - 4rr' \tan^2 \frac{m\pi}{n} = d^2,$$

which is Steiner's formula.

Exceptional case, in which the given circles touch, §§ 14, 15.

14. Let us now suppose that the given circles A, B touch internally as in fig. (12). Then if a series of circles (1, 2, 3, ...) be drawn cutting A, B at angles α, β respectively, and each of its neighbours in the series at the angle ϕ_3 we have by (6) for the radius of the circle (n),

$$\frac{1}{r_n} = M + Nn + n^2 \frac{1 + \cos\phi}{\cos\alpha - \cos\beta} \left(\frac{1}{r} - \frac{1}{r'}\right) \text{......(17)};$$

and proceeding as in § 7, we easily find that

$$M = \frac{1}{r_n} - \frac{2}{R} \cos\tfrac{1}{2}\phi - K,$$

$$N = \frac{2}{R} \cos\tfrac{1}{2}\phi - 2K,$$

where $$K = \frac{1 + \cos\phi}{\cos\alpha - \cos\beta} \left(\frac{1}{r} - \frac{1}{r'}\right).$$

Hence we have

$$\frac{1}{r} = \frac{1}{r_1} + \frac{2(n-1)}{R} \cos\tfrac{1}{2}\phi + K(n-1)^2.$$

Thus we can draw two infinite series of circles, starting with the first in any position; the radius of the $(n+1)^{\text{th}}$ circle being given by the formula

$$\frac{1}{r_{n+1}} = \frac{1}{r_1} \pm \frac{2n}{R} \cos \tfrac{1}{2}\phi + n^2 \frac{1+\cos\phi}{\cos\alpha + \cos\beta} \left(\frac{1}{r} - \frac{1}{r'}\right).$$

15. This formula may be expressed in a slightly simpler form; we know that (see *Quarterly Journal*, vol. XXI. p. 13)

$$R = \frac{r_1 r r'}{2\Delta} (4S)^{\frac{1}{2}},$$

where Δ denotes the area of the triangle formed by the centres of the circles 1, A, B; so that, if d denote the distance between the centres of A and B, p the perpendicular from the centre of (1) on the axis of A, B, we have

$$\frac{1}{r_{n+1}} = \frac{1}{r_1} \pm \frac{2n}{R} \cos^2 \tfrac{1}{2}\phi - 2n^2 \frac{r_1}{p.R} \cos^2 \tfrac{1}{2}\phi \text{......(18)}.$$

If, however, the centres of $(1, A, B)$ lie on a straight line, our two systems of circles are similar, and the radius of any circle of either system is given by

$$\frac{1}{r_{n+1}} = \frac{1}{r_1} - n^2 \frac{1+\cos\phi}{\cos\alpha - \cos\beta} \frac{d}{rr'}.$$

In the particular case when $\alpha = \pi$, $\beta = 0$, $\phi = 0$, we have

$$r_{n+1} = \frac{r_1 r r'}{rr' + n^2 r_1^2},$$

which agrees with a result previously obtained (*Quarterly Journal*, vol. XXI. p. 21).

Extension to spherical geometry, §§ 16–18.

16. For circles on the surface of a sphere the formula

$$\pi \begin{pmatrix} 1, & 2, & 3, & 4, & 5 \\ 6, & 7, & 8, & 9, & 10 \end{pmatrix} = 0$$

still holds between the powers of the circles; but for two circles (1, 2)

$$\pi_{1,2} = \tan r_1 \tan r_2 \cos \phi_{1,2};$$

and, if θ denote the imaginary circle at infinity,

$$\pi_{\theta,1} = 1, \qquad \pi_{\theta,\theta} = 1.$$

17. We shall find instead of (5)

$$\cot r_n = M\cos n\psi + N\sin n\psi - (\cos\alpha + \cos\beta\cos\omega)\operatorname{cosec}^2\omega\cot r$$
$$- (\cos\beta + \cos\alpha\cos\omega)\operatorname{cosec}^2\omega\cot r',$$

where $$4S(1-\cos\psi) + \sin^2\omega\,(1+\cos\phi) = 0.$$

And our results may be put into a real form by putting

$$\cos(r-r') = \cos d - 2\sin r\sin r'\tan^2\varpi,$$

or $$\cos\omega + 1 + 2\tan^2\varpi = 0,$$

as before.

18. The formulæ for the exceptional case when the given circle touch may be adapted in the same manner. For instance, if the centres of the first of the series lie on the great circle which passes through the poles of A, B, and $\alpha = \pi$, $\beta = \phi = 0$, we shall have

$$\cot r_{n+1} = \frac{\cos r_1\sin r\sin r' + n^2\sin^2 r_1}{\sin r_1\sin r\sin r'}.$$

SUMNER LINES ON MERCATOR'S CHART.

By *A. G. Greenhill, M.A.*

1. When the altitude of the sun or of a star is observed by the sextant, and the time by chronometer is simultaneously observed, the locus of the observer is a small circle on the terrestrial globe, of which the sun or star is in the zenith of the pole of the small circle and the angular radius of the small circle is the complement of the observed altitude.

Another such observation will give another small circle, and the position of the observer will therefore be at one of the points of intersection of the two small circles, and it is in general easy to choose the point to be taken.

If the observations are taken at sea, then an allowance for the ship's run between the two observations will indicate the positions of the ship on the small circles.

This method of determination of a ship's place at sea is due to Captain Sumner of Boston, Mass., and is now much used by navigators.

The chief difficulty, however, of the method is from the observations to draw on Mercator's chart the curve, or a small part of it in the assumed neighbourhood of the observer, corresponding to the small circle on the terrestrial globe, such a curve being called a Sumner line, which is therefore the projection of a small circle on Mercator's chart.

The numerical application of Sumner's method has been given by Sir W. Thomson in his *Tables for Facilitating Sumner's Method at Sea.*

2. Now, if θ, ϕ denote the latitude and longitude, in circular measure, of a point whose coordinates are x, y on Mercator's chart, then

$$x = c\phi, \quad y = c \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta),$$

where c denotes some constant, defining the scale of the chart, and the axis of x is in the equator.

Therefore

$$\phi = x/c, \qquad \theta = \operatorname{gd} y/c,$$

employing Cayley's notation for the *Gudermannian function;* and with *hyperbolic functions* this is equivalent to

$$\cos\theta \cosh y/c = 1,$$

or

$$\cos\theta = \operatorname{sech} y/c,$$

$$\sin\theta = \tanh y/c,$$

$$\tan\theta = \sinh y/c,$$

or

$$\tan\tfrac{1}{2}\theta = \tanh\tfrac{1}{2}y/c.$$

3. This is proved from the consideration that in order that there should be no distortion of corresponding elements of area on the sphere and on the chart, the bearing by compass of adjacent points on the sphere and on the chart should be the same, or corresponding angles should be the equal; and therefore

$$\frac{dy}{dx} = \frac{d\theta}{\cos\theta\, d\phi}.$$

Therefore, if $x = c\phi,$

$$\frac{dy}{d\theta} = c \sec\theta,$$

or

$$y = c\int_0 \sec\theta\, d\theta,$$

$$= c \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta),$$

or

$$= c \operatorname{gd}^{-1}\theta,$$

using Clifford's notation, *Collected Works*, p. 563, so that

$$\theta = \operatorname{gd} y/c.$$

4. The equation of a *loxodrome* or *rhumb line* on Mercator's chart cutting the meridians at a constant angle α is of the form

$$y = x \cot \alpha,$$

being the course described with a constant bearing by compass; and then the equation of the corresponding curve on the sphere is

$$\theta = \operatorname{gd} n\phi,$$

where

$$n = \cot \alpha;$$

equivalent to

$$\cos\theta \cosh n\phi = 1,$$

or

$$\sin\theta = \tanh n\phi,$$

$$\cos\theta = \operatorname{sech} n\phi,$$

$$\tan\theta = \sinh n\phi,$$

or

$$\tan \tfrac{1}{2}\theta = \tanh \tfrac{1}{2} n\phi.$$

5. Now, if Δ denotes the sun or star's declination, α the observed altitude, and ϕ the difference of longitude of the observer and the sun or the star, then (fig. 13)

$$PS = \tfrac{1}{2}\pi - \Delta, \quad SZ = \tfrac{1}{2}\pi - \alpha, \quad PZ = \tfrac{1}{2}\pi - \theta,$$

and

$$SPZ = \pi - \phi,$$

S denoting the sun, Z the zenith of the observer, and P the pole of the earth's axis.

Thus, from the spherical triangle SPZ,

$$\sin\alpha = \sin\Delta \sin\theta - \cos\Delta \cos\theta \cos\phi,$$

the equation connecting θ the latitude and ϕ the longitude of any point Z on the small circle; and therefore the equation of the corresponding Sumner line on the Mercator's chart is

$$\sin\alpha = \sin\Delta \tanh\frac{y}{c} - \cos\Delta \operatorname{sech}\frac{y}{c} \cos\frac{x-a}{c},$$

or

$$\cos\Delta \cos\frac{x-a}{c} = \sin\Delta \sinh\frac{y}{c} - \sin\alpha \cosh\frac{y}{c},$$

a denoting the longitude on the chart of the place where the sun or star is in the zenith.

We must distinguish two cases, according as the observed altitude α is less or greater than the declination Δ.

6. In the first case, the small circle, the locus of Z, encloses the pole P, and where the observed altitude α is less than the declination Δ the equation may be written

$$\sinh\frac{y-b}{c} = \tan\beta\cos\frac{x-a}{c} \ldots\ldots\ldots\ldots\ldots \text{(i)},$$

where

$$\cosh\frac{b}{c} = \tan\beta\tan\Delta,$$

$$\sinh\frac{b}{c} = \tan\beta\frac{\sin\alpha}{\cos\Delta},$$

and then b is the latitude of the parallel through the points B, B', where the Sumner line cuts the prime vertical of the sun, and β is the angle at which the Sumner line cuts this parallel.

The corresponding Sumner line will then be an open undulating curve (fig. 14).

As a particular case, if $\alpha = 0$, the small circle, the locus of Z, becomes a great circle, the equation of which on Mercator's chart is

$$\sinh\frac{y}{c} = \tan\beta\cos\frac{x-a}{c},$$

supposing the great circle cuts the equator at an angle β.

When the bearing is taken by compass of a mountain a great distance off, a correction must be made for the curvature of the line on the chart which is the projection of the great circle containing the line of sight.

7. In the second case the small circle, the locus of Z does not enclose the pole P, and the observed altitude α is greater than the declination Δ; the equation of the Sumner line can now be written

$$\cosh\frac{y-b}{c} = \sec\beta\cos\frac{x-a}{c} \ldots\ldots\ldots\ldots \text{(ii)},$$

where

$$\cosh\frac{b}{c} = \sec\beta\frac{\sin\alpha}{\cos\Delta},$$

$$\sinh\frac{b}{c} = \sec\beta\tan\Delta;$$

but now $SPZ = \phi$, and b is the latitude of the parallel on the chart through the points A, A' corresponding to those points on the sphere where the tangents from the pole touch the small circle, and 2β denotes the angle between these tangent lines or meridians (fig. 14).

The corresponding Sumner line will then be a closed oval curve, bounded by meridians of longitude a distance $2c\beta$ apart (fig. 15), (Sir W. Thomson, *Navigation*, p. 52), or, strictly speaking, a periodic series of such ovals will be the complete curve.

8. In the separating case, when the observed altitude is equal to the declination, the small circle passes through the pole, and the equation of the corresponding Sumner line is

$$\cot\Delta \cos\frac{x-a}{c} = \cosh\frac{y}{c} - \sinh\frac{y}{c}$$

$$= \exp\left(-\frac{y}{c}\right),$$

or

$$\frac{y-b}{c} = \log\sec\frac{x-a}{c},$$

the equation of the catenary of equal strength; this may also be written

$$\tanh\frac{y-b}{2c} = \tan^2\frac{x-a}{2c}.$$

9. Knowing approximately the latitude, the Sumner line will determine the longitude with the greatest accuracy when the Sun or star observed is on or as near as possible to the prime vertical, since the corresponding Sumner line then cuts the parallel of latitudes at a maximum angle.

10. It is a curious property of these Sumner lines that the arcs of the curves are elliptic integrals of the first kind.

For taking equation (i), and differentiating it,

$$\frac{dy}{dx} = \frac{-\tan\beta\sin\dfrac{x-a}{c}}{\cosh\dfrac{y-b}{c}} = \frac{-\tan\beta\sin\dfrac{x-a}{c}}{\sqrt{\left(1+\tan^2\beta\cos^2\dfrac{x-a}{c}\right)}},$$

$$\frac{ds}{dx} = \frac{\sec\beta}{\sqrt{\left(1+\tan^2\beta\cos^2\dfrac{x-a}{c}\right)}} = \frac{1}{\sqrt{\left(1-\sin^2\beta\sin^2\dfrac{x-a}{c}\right)}},$$

an elliptic integral of the first kind; so that

$$\sin\frac{x-a}{c}=\operatorname{sn}\frac{s}{a},$$

$$\sinh\frac{y-b}{c}=\tan\beta\operatorname{cn}\frac{s}{c},$$

$$\cosh\frac{y-b}{c}=\sec\beta\operatorname{dn}\frac{s}{c},$$

the modular angle being β.

14. Again from equation (ii), differentiating,

$$\frac{dy}{dx}=\frac{-\sec\beta\sin\dfrac{x-a}{c}}{\sqrt{\left(\sec^2\beta\cos^2\dfrac{x-a}{c}-1\right)}},$$

$$\frac{ds}{dx}=\frac{\tan\beta}{\sqrt{\left(\sec^2\beta\cos^2\dfrac{x-a}{c}-1\right)}}$$

$$=\frac{\sin\beta}{\sqrt{\left(\sin^2\beta-\sin^2\dfrac{x-a}{c}\right)}},$$

an elliptic integral of the first kind; so that

$$\sin\frac{x-a}{c}=\sin\beta\operatorname{sn}\frac{s}{c},$$

$$\cos\frac{x-a}{c}=\operatorname{dn}\frac{s}{c},$$

$$\cosh\frac{y-b}{c}=\sec\beta\operatorname{dn}\frac{s}{c},$$

the modular angle being β.

For curves whose arcs are elliptic integrals of the first kind, consult Kiepert, *Crelle*, 79.

Comparing these expressions with the well known expressions for the finite oscillations of a simple circular pendulum, we notice that the arc s in a Sumner line is proportional to the time t, when the longitude ϕ or x/c is the angle made by the pendulum with the vertical.

ON THE SOLUTION OF LEGENDRE'S EQUATION IN A PARTICULAR CASE.

By *A. R. Forsyth.*

PROF. CAYLEY has pointed out to me an error in my book on differential equations which occurs in the solution of Legendre's equation. It is, in effect, there stated (p. 150) that, in the case when $2n$ is an odd positive integer, two independent integrals are given by

$$y_2 = x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \dots$$

and

$$y = \tfrac{1}{2}y_2 \log\left(\frac{x+1}{x-1}\right) - \left\{x^{-n-2} + \frac{(n+2)(n+3)(3n+4)}{3(2n+3)(2n+4)}x^{-n-4} + \dots\right\}$$

$$= \tfrac{1}{2}y_2 \log\left(\frac{x+1}{x-1}\right) - w_1\text{*};$$

and similarly for the case when $2n$ is an odd negative integer. "Now" (I quote from Prof. Cayley's letter) "take n a
"general fraction, so that we have the solutions

$$y_1 = x^n - \text{etc.}, \qquad y_2 = x^{-n-1} + \text{etc.};$$

"then $\quad \tfrac{1}{2}y_2 \log\left(\frac{x+1}{x-1}\right) - w_1 = y_2\left(\frac{1}{x} + \frac{1}{3x^3} + \dots\right) - w_1,$

"quà solution, must be equal to $Cy_1 + Dy_2$. But it contains
"only terms x^{-n-2}, x^{-n-4}, ...; hence there can be no term Cy_1;
"and clearly it is not $= Dy_2$; whence it must be identically
"$= 0$, and it is easy for a few terms to verify that this is so.
"And being in general $= 0$, there is nothing to interfere
"herewith for the particular value $n = r + \frac{1}{2}$, *i.e.* for this
"value $\frac{1}{2}y_2 \log\left(\frac{x+1}{x-1}\right) - w_1 = 0$, and your solution really is
"$= Ay_2$. Similarly for $n = -r - \frac{1}{2}$... ."

The solution corrected is given in the following note, which will replace §§ 93–95 of my book, and the paragraphs of which are numbered accordingly. It will be seen that the present § 95 is practically the same as the former §§ 93 and 94; it is needed for subsequent reference. And to make the articles complete I have given the example which appears in the book at the end of § 95.

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[*] There is an obvious mistake in the sign of w_1 in the book.

93. It therefore remains to obtain the primitive in the last two cases.

Consider first the case of $2n$ equal to an odd positive integer; then

$$y_2 = x^{-n-1} + \frac{(n+1)(n+2)}{2\,(2n+3)}\,x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\,.\,4\,(2n+3)\,(2n+5)}\,x^{-n-5} + \ldots$$

is a definite solution, and we have to find a second and different particular solution. In the first instance, assume

$$2n = 2p + 1 + \theta,$$

where θ is an infinitesimal quantity which will ultimately be made zero. Then, so long as θ is not zero, the quantity

$$y_1 = x^n - \frac{n\,(n-1)}{2\,(2n-1)}\,x^{n-2} + \ldots$$

is also a definite solution; and it ceases to be so by the vanishing of θ, since θ enters as a factor into the denominator of the coefficient of x^{n-2p-2} and all lower powers. Now we have

$$\begin{aligned} Ay_1 = A\Big\{x^n &- \frac{n\,(n-1)}{2\,(2n-1)}\,x^{n-2} + \ldots \\ &+ (-1)^p\,\frac{n\,(n-1)\ldots(n-2p+1)}{2.4\ldots 2p\,(2n-1)\,(2n-3)\ldots(2n-2p+1)}\,x^{n-2p} \\ &+ (-1)^{p+1}\,\frac{n\,(n-1)\ldots(n-2p-1)}{2.4\ldots(2p+2)(2n-1)(2n-3)\ldots(2n-2p-1)}\,x^{n-2p-2} \\ &+ (-1)^{p+2}\,\frac{n\,(n-1)\ldots(n-2p-3)}{2.4\ldots(2p+4)(2n-1)(2n-3)\ldots(2n-2p-3)}\,x^{n-2p-4} + \ldots\Big\} \\ = A\Big\{x^n &- \frac{n\,(n-1)}{2\,(2n-1)}\,x^{n-2} + \ldots \\ &+ (-1)^p\,\frac{n\,(n-1)\ldots(n-2p+1)}{2\,.\,4\ldots 2p\,(2n-1)\,(2n-3)\ldots(2n-2p+1)}\,x^{n-2p}\Big\} \\ &+ \frac{Cx^{n-2p-2}}{2n-2p-1}\left\{1 - \frac{(n-2p-2)\,(n-2p-3)}{(2p+4)\,(2n-2p-3)}\,x^{-2} + \ldots\right\}, \end{aligned}$$

where

$$C = (-1)^{p+1} A\,\frac{n\,(n-1)\ldots(n-2p-1)}{2\,.\,4\ldots(2p+2)\,(2n-1)\,(2n-3)\ldots(2n-2p+1)},$$

and so is determinate and finite. But

$$n - 2p - 2 = -(n+1) + \theta,$$

and therefore

$$x^{n-2p-2} = x^{-(n+1)}\,.\,x^{\theta} = x^{-n-1}\,(1 + \theta\log x).$$

Also the coefficient of x^{-2r} within the second bracket is

$$(-1)^r \frac{(n-2p-2)(n-2p-3)\dots(n-2p-2r-1)}{(2p+4)(2p+6)\dots(2p+2r+2)(2n-2p-3)(2n-2p-5)\dots(2n-2p-2r-1)},$$

i.e. is
$$(-1)^r \frac{(n+1-\theta)(n+2-\theta)\dots(n+2r-\theta)}{(2n+3-\theta)(2n+5-\theta)\dots(2n+2r+1-\theta)(\theta-2)(\theta-4)\dots(\theta-2r)},$$

i.e., is
$$\frac{(n+1-\theta)(n+2-\theta)\dots(n+2r-\theta)}{(2n+3-\theta)(2n+5-\theta)\dots(2n+2r+1-\theta)(2-\theta)(4-\theta)\dots(2r-\theta)},$$

i.e., is
$$\frac{(n+1)(n+2)\dots(n+2r)}{(2n+3)(2n+5)\dots(2n+2r+1)\,2.4.6\dots..2r}(1+C_r\theta),$$

where
$$C_r = \sum_{s=1}^{s=r}\frac{1}{2s} + \sum_{s=1}^{s=r}\frac{1}{2n+2s+1} - \sum_{s=1}^{s=2r}\frac{1}{n+s}.$$

Hence

$$Ay_1 = A\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \dots + (-1)^p \frac{n(n-1)\dots(n-2p+1)}{2.4\dots2n(2n-1)(2n-3)\dots(2n-2p+1)}x^{n-2p}\right\}$$
$$+ Cx^{-(n+1)}\frac{1+\theta\log x}{\theta}\left[1+\sum_{r=1}^{r=\infty}\left\{\frac{(n+1)(n+2)\dots(n+2r)}{2.4\dots2r(2n+3)(2n+5)\dots(2n+2r+1)}(1+C_r\theta)\,x^{-2r}\right\}\right].$$

When the second part of the right-hand side is expanded the aggregate of terms which involve $\frac{1}{\theta}$ is

$$\frac{C}{\theta}y_2;$$

the aggregate of terms which involve $\log x$ is

$$Cy_2 \log x;$$

and there remains the aggregate of terms independent of θ (and also as it appears of $\log x$), as well as a further aggregate of terms multiplied by positive powers of θ, most of which have been omitted and all of which disappear when θ is made to vanish. From the first part of the right-hand side there is an aggregate of terms independent of θ, as well as an aggregate of terms which disappear when θ is made zero. Hence the primitive of the equation is

$$\begin{aligned} y &= By_2 + Ay_1 \\ &= \left(B + \frac{C}{\theta}\right) y_2 + C\left(y_2 \log x + T_n + R_n\right) \\ &= Dy_2 + C\left(y_2 \log x + T_n + R_n\right), \end{aligned}$$

on changing the arbitrary constants. Here T_n stands for

$$\frac{A}{C}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \dots \right.$$
$$\left. + (-1)^p \frac{n(n-1)\dots(n-2p+1)}{2.4\dots 2p\,(2n-1)(2n-3)\dots(2n-2p+1)}\chi^{n-2p}\right\},$$

and R_n stands for

$$x^{-n-1}\sum_{r=1}^{r=\infty}\left\{\frac{(n+1)(n+2)\dots(n+2r)}{2.4\dots 2r\,(2n+3)(2n+5)\dots(2n+2r+1)}C_r x^{-2r}\right\},$$

the value of C_r being

$$\sum_{s=1}^{s=r}\left(\frac{1}{2s} + \frac{1}{2n+2s+1} - \frac{1}{n+2s-1} - \frac{1}{n+2s}\right).$$

The value of the coefficient A/C which occurs in T_n is

$$\left\{\frac{4.8.12\dots.(4n-2)}{1.3.5\dots.2n}\right\}^2 (8n+4),$$

so that we may write T_n in the form

$$\left\{\frac{4.8.12...(4n-2)}{1.3.5...2n}\right\}^2 (8n+4)\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2}+\dots\right.$$
$$\left. - n(n-1)\left\{\frac{\frac{1}{2}.\frac{3}{2}...(n-2)}{2.4...(2n-1)}\right\}^2 x^{-n+1}\right\}.$$

The second particular solution of the equation is thus

$$y_2 \log x + T_n + R_n;\text{*}$$

and it will be noticed that that part of it which is expansible in descending powers of x begins with a term involving x^{+n} and contains no term involving x^{-n-1}.

94. Consider now the case of $2n$ equal to an odd negative integer; the integral y_1 is definite, but

$$y_2 = x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}+\dots$$

will then, except for one value of n, viz., $-\frac{1}{2}$, not be a definite solution. But for the case of $n=-\frac{1}{2}$ the primitive is

$$y = Ay_1 + By_2,$$

and so does not come under the exception considered; and, for the present purpose, $-2n$ may be 3 or some greater odd integer.

Before assuming n to be half an odd integer, write

$$-n = m+1$$

(so that $2m$ is a positive odd integer when the assumption as to the special value of n is made). Then

$$y_1 = x^{-m-1} + \frac{(m+1)(m+2)}{2(2m+3)}x^{-m-3}+\dots$$
$$= Y_2,$$

and
$$y_2 = x^m - \frac{m(m-1)}{2(2m-1)}x^{m-2}+\dots$$
$$= Y_1,$$

---

* The solution thus given corresponds to that for Bessel's equation, Ex. 1, p. 165, due to Hankel.

where Y_1 and Y_2 are the special solutions of

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+m(m+1)y=0,$$

m being positive. When $2m$ is an odd positive integer we know from the preceding investigation that the primitive of this is

$$y=BY_2+A(Y_2\log x+T_m+R_m),$$

where

$$T_m=\left\{\frac{4.8.12\ldots(4m-2)}{1.3.5\ldots 2m}\right\}^2(8m+4)\left[x^m-\frac{m(m-1)}{2(2m-1)}x^{m-2}+\ldots\right.$$
$$\left.-m(m-1)\left\{\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots(m-2)}{2.4\ldots(2m-1)}\right\}^2x^{-m+1}\right],$$

and

$$R_m=x^{-m-1}\sum_{r=1}^{r=\infty}\left\{\frac{(m+1)(m+2)\ldots(m+2r)}{2.4\ldots 2r(2m+3)(2m+5)\ldots(2m+2r+1)}A_rx^{-2r}\right\},$$

the value of A_r being

$$\sum_{s=1}^{s=r}\left(\frac{1}{2s}+\frac{1}{2m+2s+1}-\frac{1}{m+2s-1}-\frac{1}{m+2s}\right).$$

Hence the primitive of

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0,$$

in the case when $2n$ is an odd negative integer other than -1, is

$$y=By_1+A(y_1\log x+V_n+U_n),$$

where

$$y_1=x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}-\ldots$$

$$V_n=\left\{\frac{4.8.12\ldots(-4n-6)}{1.3.5\ldots(-2n-2)}\right\}^2(-8n-4)\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}+\ldots\right.$$
$$\left.-(n+1)(n+2)\left\{\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots(-n-3)}{2.4\ldots(-2n-3)}\right\}^2x^{n+2}\right],$$

and

$$U_n=x^n\sum_{r=1}^{r=\infty}\left\{\frac{n(n-1)\ldots(n-2r+1)}{2.4\ldots 2r(2n-1)(2n-3)\ldots(2n-2r+1)}(-1)^rE_rx^{-2r}\right\},$$

where in U_n the value of E_r is

$$\sum_{s=1}^{s=r}\left(\frac{1}{2s}+\frac{1}{2s-2n-1}-\frac{1}{2s-n-2}-\frac{1}{2s-n-1}\right).$$

The second particular solution of the equation in this case is thus

$$y_1\log x+V_n+U_n;$$

and it will be noticed that that part of it which is expansible in descending powers of x begins with a term involving x^{-n-1} and contains no term involving x^n.

Since $2n$ is an odd integer, the equation can be written

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+(p^2-\tfrac{1}{4})y=0,$$

where p is an integer.

The case of p positive is that considered in § 93; the case of p negative is that considered in § 94; and the case of p zero is not in any way exceptional. Properties of the functions defined by the differential equation in the present form have been discussed by Mr. W. M. Hicks in his memoir on "Toroidal Functions," *Phil. Trans. Roy. Soc.* (1881), pp. 609–652.

Ex. 1. Assuming the result of Ex. 1 in § 64, shew how the solution of

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}=\tfrac{1}{4}y$$

can be derived from that of

$$(1-k^2)\frac{d^2v}{dk^2}+\frac{1-3k^2}{k}\frac{dv}{dk}=v,$$

which is the differential equation for the quarter-period in elliptic functions.

Ex. 2. Prove that the Particular Integral of the equation

$$(1-x^2)\frac{d^2w}{dx^2}+n(n+1)w=\frac{dP_n}{dx}$$

is λP_{n-1}, where λ is a constant; and that the Particular Integral of the equation

$$(1-x^2)\frac{d^2w}{dx^2}+n(n+1)w=\frac{dQ_n}{dx}$$

is $\lambda' Q_{n+1}$, where λ' is a constant.

95. In the general case of the differential equation, as represented by I, II, III of § 92, it is possible to express the second particular solution in terms of that already obtained and similar of functions. Let v denote the particular solution already obtained, so that for instance v would be P_n in I; and let

$$y = uv - w,$$

where u and w are as yet indeterminate. When this is substituted in the differential equation, we have

$$-\left[\frac{d}{dx}\left\{(1-x^2)\frac{dw}{dx}\right\} + n(n+1)w\right] + v\left\{(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx}\right\}$$
$$+ 2\frac{du}{dx}(1-x^2)\frac{dv}{dx} + u\left[\frac{d}{dx}\left\{(1-x^2)\frac{dv}{dx}\right\} + n(n+1)v\right] = 0.$$

Since v is a solution, the last term disappears; and as the only condition imposed on u and w is that y must satisfy the equation, we may arbitrarily assign another. Choosing this so that the coefficient of v may vanish, we have

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} = 0,$$

and therefore

$$(x^2-1)\frac{du}{dx} = \text{constant}.$$

As we are seeking a particular solution, it is convenient to have it as simple as possible; and therefore, giving a special value to the constant, we may write

$$(x^2-1)\frac{du}{dx} = -1,$$

so that a value of u is given by

$$u = \tfrac{1}{2}\log\left(\frac{x+1}{x-1}\right).$$

The equation to determine w now becomes

$$\frac{d}{dx}\left\{(1-x^2)\frac{dw}{dx}\right\} + n(n+1)w = 2\frac{dv}{dx}.$$

When the Particular Integral, say w_1, of this is obtained, the second solution of the original equation is

$$y = \tfrac{1}{2} v \log \left(\frac{x+1}{x-1}\right) - w_1.$$

The value of w_1 as a series of descending powers of x is easily obtained. Thus in the case when n is a positive integer we take

$$v = x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} - \ldots,$$

and at once have the equation, which determines w_1, in the form

$$\frac{d}{dx}\left\{(1-x^2)\frac{dw_1}{dx}\right\} + n(n+1) w_1 = 2n\left[x^{n-1} - \frac{(n-1)(n-2)}{2(2n-1)} x^{n-3} + \ldots\right].$$

Let $$w_1 = C_1 x^{n-1} + C_2 x^{n-3} + C_3 x^{n-5} + \ldots;$$

then, substituting and equating the coefficients of the highest term, we have

$$C_1 \{n(n+1) - n(n-1)\} = 2n,$$

or $$C_1 = 1;$$

and equating the coefficients of the terms involving x^{n-2r+1}, we have

$$C_r \{n(n+1) - (n-2r+1)(n-2r+2)\} + (n-2r+3)(n-2r+2) C_{r-1}$$
$$= (-1)^{r-1} 2 \frac{n(n-1)(n-2)\ldots(n-2r+2)}{2.4\ldots(2r-2)(2n-1)\ldots(2n-2r+3)}.$$

The general value of C_r, deducible from this, is complicated; the values of the earlier coefficients are

$$C_2 = -\frac{(n-1)(n-2)(3n-1)}{3(2n-1)(2n-2)},$$

$$C_3 = \frac{(n-1)(n-2)(n-3)(n-4)(30n^2 - 50n + 12)}{3.4.5(2n-1)(2n-2)(2n-3)(2n-4)},$$

and so on; but there is no advantage in writing down more of the coefficients, as the expression for w_1 will soon be put into a different form.

ON THE TRISECTION OF THE PERIODS FOR WEIERSTRASS'S ELLIPTIC FUNCTIONS.

By *William Burnside, M.A.*

THE triplication formula for $P(u)$ may be most symmetrically obtained from the equation

$$\frac{\sigma(3u)}{\sigma^9(u)} = \frac{1}{4}\begin{vmatrix} P'(u), & P''(u) \\ P''(u), & P'''(u) \end{vmatrix} \text{............}(1),$$

where $$P(u) = -\frac{d^2}{du^2}\{\log\sigma(u)\},$$

and $$P'(u) = -\{4P^3(u) - g_2P(u) - g_3\}^{\frac{1}{2}};$$

$$\begin{vmatrix} P'(u), & P''(u) \\ P''(u), & P'''(u) \end{vmatrix} = 12P(4P^3 - g_2P - g_3) - (6P^2 - \tfrac{1}{2}g_2)^2$$
$$= 12P^4 - 6g_2P^2 - 12g_3P - \tfrac{1}{4}g_2^2;$$

and hence from (1)

$$9P(3u) - 9P = -\frac{d^2}{du^2}\log(3P^4 - \tfrac{3}{2}g_2P^2 - 3g_3P - \tfrac{1}{16}g_2^2)\text{...}(2).$$

Writing for a time D in the place of

$$3P^4 - \tfrac{3}{2}g_2P^2 - 3g_3P - \tfrac{1}{16}g_2^2,$$

(2) becomes, on performing the differentiation,

$$P(3u) - P = -\frac{(P')^2}{D^2}(2P^6 - \tfrac{5}{2}g_2P^4 - 10g_3P^3 - \tfrac{5}{8}g_2^2P^2$$
$$- \tfrac{1}{2}g_2g_3P + \tfrac{1}{32}g_2^3 - g_3^2)\text{...}(3).$$

Writing e_λ for any one of the roots of

$$4x^3 - g_2x - g_3 = 0,$$

$$\frac{P(3u) - e_\lambda}{P - e_\lambda} = 1 - 4\,\frac{P^2 + e_\lambda P + \dfrac{g_3}{4e_\lambda}}{D^2}(2P^6 - \ldots + \tfrac{1}{32}g_2^2 - g_3^2).$$

After a certain amount of algebraical reduction, this equation becomes

$$\frac{P(3u) - e_\lambda}{P - e_\lambda}$$
$$= \left\{\frac{P^4 - 4e_\lambda P^3 - (12e_\lambda^2 - \tfrac{3}{2}g_2)P^2 - 4e_\lambda^3P - 8e_\lambda^4 + 3e_\lambda^2g_2 + \tfrac{3}{4}g_2^2}{D}\right\}^2\text{...}(4),$$

which seems the simplest form of the triplication equation.

If the right-hand side of (4) be written

$$\frac{N^2}{D^2},$$

it is easy to shew that

$$D = 3(P - e_\lambda)^4 + 12e_\lambda (P - e_\lambda)^3 + 18\alpha_\lambda (P - e_\lambda)^2 - 9\alpha_\lambda^2,$$

and $N = (P - e_\lambda)^4 - 18\alpha_\lambda (P - e_\lambda)^2 - 36e_\lambda \alpha_\lambda (P - e_\lambda) - 27\alpha_\lambda^2,$

where $\alpha_\lambda = e_\lambda^2 - \frac{1}{12} g_2 = \frac{1}{3}(e_\lambda - e_\mu)(e_\lambda - e_\nu)$;

and hence that if

$$D = f(P - e_\lambda),$$

then $$N = \frac{(P - e_\lambda)^4}{9\alpha_\lambda^2} f\left(\frac{3\alpha_\lambda}{P - e_\lambda}\right).$$

As the last result suggests, (4) might have been obtained from the relation

$$P(u + \omega) - e_\lambda = \frac{3\alpha_\lambda}{P - e_\lambda}.$$

For the trisection of the periods the equations

$$P(3u) = \infty, \qquad P(3u) - e_\lambda = 0,$$

must be solved; and the above relation between the quantities N and D shews that the roots of the three included in the latter form are rational algebraical functions of the roots of the first and the quantities e_λ. The equation $P(3u) = \infty$ expressed in terms of P is

$$P^4 - \tfrac{1}{2} g_2 P^2 - g_3 P - \tfrac{1}{48} g_2^2 = 0 \text{ ....(5).}$$

If Q_2, Q_3 are positive quantities defined by the equations

$$(Q_2 \pm Q_3 i)^2 = \tfrac{1}{3}\left(g_2 - \Delta^{\frac{1}{3}} e^{\pm \frac{2}{3} i\pi}\right),$$

and Q_1 the positive or negative square root of $\frac{1}{3}(g_2 - \Delta^{\frac{1}{3}})$ according as g_3 is positive or negative, where

$$\Delta = g_2^3 - 27g_3^2,$$

then the roots of the above biquadratic are in every case

$$\tfrac{1}{2}Q_1 + Q_2, \quad \tfrac{1}{2}Q_1 - Q_2, \quad -\tfrac{1}{2}Q_1 + Q_3 i, \quad -\tfrac{1}{2}Q_1 - Q_3 i.$$

These quantities then are the values of

$$P\{\tfrac{1}{3}(2n\omega + 2n'\omega')\}$$

for various values of n and n'.

In assigning these values the cases of a positive and a negative discriminant must be taken separately.

I. $\Delta > 0$.

In this case the quantities e_1, e_2, e_3 are all real, and taking them in descending order of magnitude as written, the usual notation is to write

$$P(\omega) = P(\omega_1) = e_1, \qquad P(\omega') = P(\omega_3) = e_3,$$

ω_1 and ω_3 being respectively a positive real quantity and a positive imaginary.

The roots of (5) may accordingly be written

$$P(\tfrac{2}{3}\omega_1), \quad P(\tfrac{2}{3}\omega_3), \quad P\{\tfrac{1}{3}(2\omega_1 + 2\omega_3)\}, \quad P\{\tfrac{1}{3}(2\omega_1 - 2\omega_3)\}.$$

Of these the first two are real and such that

$$P(\tfrac{2}{3}\omega_1) > P(\tfrac{2}{3}\omega_3);$$

the latter two are conjugate imaginaries, such that as pointed out by Mr. Forsyth in an article in the *Quarterly Journal of Mathematics*,

$$P\{\tfrac{1}{3}(2\omega_1 + 2\omega_3)\} - P\{\tfrac{1}{3}(2\omega_1 - 2\omega_3)\}$$

is a negative imaginary. Hence.

$$\begin{aligned} P(\tfrac{2}{3}\omega_1) &= \tfrac{1}{2}Q_1 + Q_2, \\ P(\tfrac{2}{3}\omega_3) &= \tfrac{1}{2}Q_1 - Q_2, \\ P\{\tfrac{1}{3}(2\omega_1 + 2\omega_3)\} &= -\tfrac{1}{2}Q_1 - Q_3 i, \\ P\{\tfrac{1}{3}(2\omega_1 - 2\omega_3)\} &= -\tfrac{1}{2}Q_1 + Q_3 i. \end{aligned}$$

II. $\Delta < 0$.

In this case two of the quantities e_1, e_2, e_3 are imaginary; e_2 is taken as real, so that ω and ω' are conjugate imaginaries, and $e_1 - e_3$ is taken to be a positive imaginary.

It is then usual to write

$$P(\omega + \omega') = P(\omega_2) = e_2, \qquad \omega' - \omega = \omega'_2,$$

ω_2 and ω_2' being as before respectively a positive real quantity and a positive imaginary.

The roots of (5) may then be written

$$P(\tfrac{2}{3}\omega_2), \quad P(\tfrac{2}{3}\omega'_2), \quad P\{\tfrac{1}{3}(\omega_2 + \omega'_2)\}, \quad P\{\tfrac{1}{3}(\omega_2 - \omega'_2)\},$$

and the same considerations as in the former case shew that

$$\begin{aligned} P(\tfrac{2}{3}\omega_2) &= \tfrac{1}{2}Q_1 + Q_2, \\ P(\tfrac{2}{3}\omega'_2) &= \tfrac{1}{2}Q_1 - Q_2, \\ P\{\tfrac{1}{3}(\omega_2 + \omega'_2)\} &= -\tfrac{1}{2}Q_1 - Q_3 i, \\ P\{\tfrac{1}{3}(\omega_2 - \omega'_2)\} &= -\tfrac{1}{2}Q_1 + Q_3 i. \end{aligned}$$

In either case, as stated above, the quantities

$$P\{\tfrac{1}{3}(n\omega + n'\omega')\}$$

can for all integral values of n and n' be expressed as rational algebraical functions of those already found and of e_1, e_2, e_3. Instead of forming a table, two examples are given.

1. $\Delta > 0$, to find $P(\tfrac{1}{3}\omega_1)$,

$$\begin{aligned} P(\tfrac{1}{3}\omega_1) &= P(\tfrac{5}{3}\omega_1) = P(\omega_1 + \tfrac{2}{3}\omega_1) \\ &= e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{P(\tfrac{2}{3}\omega) - e_1} \\ &= \frac{e_1(\tfrac{1}{2}Q_1 + Q_2) + e_1^2 + e_2 e_3}{\tfrac{1}{2}Q_1 + Q_2 - e_1}. \end{aligned}$$

2. $\Delta < 0$, to find $P\{\tfrac{1}{3}(\omega_2 + 2\omega_2')\}$

$$\begin{aligned} P\{\tfrac{1}{3}(\omega_2 + 2\omega_2')\} &= P\{\omega_2' + \tfrac{1}{3}(\omega_2 - \omega_2')\} \\ &= P\{\omega_2 + \tfrac{1}{3}(\omega_2 - \omega_2')\}, \end{aligned}$$

since $\omega_2' - \omega_2$ is a period,

$$\begin{aligned} &= e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{P\{\tfrac{1}{3}(\omega_2 - \omega_2')\} - e_2} \\ &= \frac{e_2(-\tfrac{1}{2}Q_1 + Q_3 i) + e_2^2 + e_1 e_3}{-\tfrac{1}{2}Q_1 + Q_3 i - e_2}. \end{aligned}$$

It may be observed that in the case of Δ negative $P\{\tfrac{1}{6}(\omega_2 \pm \omega_2')\}$ can also be expressed: since

$$\tfrac{1}{6}(\omega_2 + \omega_2') = \tfrac{1}{2}(\omega_2 + \omega_2') - \tfrac{1}{3}(\omega_2 + \omega_2') = \omega' - \tfrac{1}{3}(\omega_2 + \omega_2')$$

and

$$\tfrac{1}{6}(\omega_2 - \omega_2') = \omega - \tfrac{1}{3}(\omega_2 - \omega_2').$$

In an important special case, when $g_2 = 0$, $g_3 = 4$, and therefore $\Delta < 0$ equation (4) becomes

$$P(3u) - 1 = (P - 1)\left[\frac{(P + 2)(P^3 - 6P^2 + 4)}{3P(P^3 - 4)}\right]^2$$

from which

$$\begin{aligned} P(\tfrac{1}{3}\omega_2) &= 2^{\frac{4}{3}} + 2 + 2^{\frac{2}{3}} \\ P(\tfrac{1}{3}\omega_2') &= 2 \\ P(\tfrac{2}{3}\omega_2) &= 2^{\frac{2}{3}} \\ P(\tfrac{2}{3}\omega_2') &= 0. \end{aligned}$$

SOME PROPERTIES OF CONFOCAL CONICS AND A DERIVED CUBIC.

By *F. Morley.*

It is fundamental in Geometrical Conics that when two tangents OP, OP_1 are drawn to a conic whose foci are S, S_1, then OS, OS_1 subtend equal or supplementary angles at P and at P_1, and OP, OP_1 subtend equal or supplementary angles at S and S_1. The chief point to be proved is that, looking on OS and OS_1 as tangents to a thin conic SS_1, the same properties belong to any two confocals; and on this is based an elementary discussion of a special circular cubic.

Since working out the analytical proof I have remembered that Smith's Conics (§ 228, p. 249) implicitly includes a good deal of the first half. But I give my proof, as it seems that the use made of the circle is of importance in the treatment of confocals.

i. If $S=0$, $S_1=0$, $S_2=0$ be 3 conics, then the derived conics $S+k_1S_1=0$, $S+k_2S_2=0$ meet on $k_1S_1=k_2S_2$, which passes through the intersections of $S_1=0$, $S_2=0$.

In particular if $S_1=0$, $S_2=0$ are circles so is $k_1S_1=k_2S_2$, and if the first two are concentric the third is concentric with them.

Hence if two lines meet two concentric circles in $ABCD$, $A_1B_1C_1D_1$, the four intersections of AA_1, DD_1, with BB_1, CC_1 lie on a concentric circle.

[Since we have two pairs of chords from each circle there are four concentric circles containing the intersections of one chord of one circle with one of the other. The radii of the four are not connected by any very simple relation. There is a relation involving the six radii irrespective of the position of the lines.]

Reciprocate with regard to the centre of the circles. Then if from two points S, S_1 without two concentric circles tangents are drawn, and if P, P_1 are two intersections of those to the outer circle, and p, p_1 of those to the inner, the four lines joining P, P_1 to p, p_1 touch a concentric circle.

ii. Now suppose A, A_1; B, B_1; C, C_1 the intersections of four tangents to a circle. Then if the circle is inscribed (fig. 16) we have

$$\begin{aligned} AB - A_1B &= \text{difference of tangents from } A, A_1 \\ &= AB_1 - A_1B_1, \\ AC - A_1C &= A_1C_1 - A\,C_1, \\ BC - B_1C &= B\,C_1 - B_1C_1, \end{aligned}$$

and therefore conics can be described with any pair as foci, passing through another pair; thus

AA_1 foci, BB_1 on a hyperbola, same branch,
CC_1 on a hyperbola, different branches.
BB_1 foci, CC_1 on a hyperbola, same branch,
AA_1 on a hyperbola, same branch.
CC_1 foci, AA_1 on an ellipse,
BB_1 on a hyperbola, same branch.

Similarly in fig. 17, when the circle is escribed,

$$\begin{aligned} A_1B - A\,B &= A\,B_1 - A_1B_1, \\ A\,C - A_1C &= A_1C_1 - A\,C_1, \\ B\,C + B_1C &= B\,C_1 + B_1C_1, \end{aligned}$$

and hence:

AA_1 foci, BB_1 on a hyperbola, different branches,
CC_1 on a hyperbola, different branches.
BB_1 foci, CC_1 on an ellipse,
AA_1 on an ellipse.
CC_1 foci, AA_1 on an ellipse,
BB_1 on a hyperbola, different branches.

Thus, as would be expected, each case occurs the same number of times; in four the points are on the same branch of a hyperbola, in four on different branches, in four on an ellipse.

Let O be the centre of the circle; then since AO makes equal angles with AB, AC, AO is a tangent. Take for instance fig. 16 and let C, C_1 be foci. Then O is the pole of AA_1. The figure shews at once that the angles ACO, A_1CO are equal; and the angle

$$AOC_1 = BAO - BC_1O = \tfrac{1}{2}(BAB_1 - BC_1B_1) = \tfrac{1}{2}AB_1C = A_1OC.$$

So that the figure gives a ready proof of the facts we assume.

iii. By § 2, the points P, P_1 of § 1 lie on a conic whose foci are S, S_1. And if O is the common centre of the three circles, O is the pole of PP_1. Also p, p_1 lie on a confocal conic, and O is the pole of pp_1.

Hence the result of § 1 gives that if tangents OP, OP_1 and Op, Op_1 be drawn from a point to two confocal conics, O will be equidistant from the lines joining P, P_1 to p, p_1. And since two tangents make equal angles with the line to the centre, Op, Op_1 subtend equal or supplementary angles at P and at P_1, and OP, OP_1 subtend equal or supplementary angles at p and at p_1.

iv. In fact, since the four such lines as Pp touch a circle, a conic with foci PP_1 will go through p, p_1 and touch Op, Op_1, so that a conic can be described with any two points on a conic as foci and having double contact with a confocal, or: if a conic has a fixed focus S and has double contact with a fixed conic, the locus of the other focus is the two confocals through S. The chord of contact will pass through one of two fixed conjugate points.

v. We see that if two conics have double contact the lines joining the foci of the one to the foci of the other touch a circle.

Now Mr. H. W. Richmond has pointed out to me that if A, B and C, D be foci of two conics, the four lines AC, AD, BC, BD and the four common tangents touch a conic. His method of proof is as follows: let α, β, γ, δ be the perpendiculars from A, B, C, D on any line, then the tangential equations of the conics are $\alpha\beta = k$, $\gamma\delta = k_1$, and hence their common tangents touch $k_1\alpha\beta = k\gamma\delta$, to which $\alpha = 0$, $\gamma = 0$ is also a tangent, as well as the three other such lines.

Hence if U_1, U_2 be conics having double contact, C the circle touched by lines joining the foci, V_1 a confocal of U_1, and V_2 a confocal of U_2, such that one common tangent of V_1 and V_2 touches C, then the other three common tangents also touch C.

vi. Let $OPSQ$, $OP_1S_1Q_1$ be focal chords of a conic. Then, by the converse of § 2, SP, SP_1, S_1P, S_1P_1 touch a circle. If SP_1 and S_1P cut at O_1, O and O_1 lie on a conic whose foci are SS_1. Hence the four intersections of SP_1, SQ_1 with S_1P, S_1Q lie on the confocals through O, and by considering the figure it is clear that two lie on each confocal (fig. 18).

Obviously the poles of the lines joining P, Q to P_1, Q_1 lie on the lines bisecting the angle at O, two on each

vii. The locus of the points of contact of tangents from a fixed point O to a family of confocals is the same as the locus of the feet of normals, since the tangent to one confocal is the normal to the other passing through the point of contact. The locus is known to be a circular cubic (compare Higher Plane Curves, § 164); clearly it touches the two confocals through O, and hence O is a node at which the tangents are at right angles. Now, if OP, OQ are chords making equal angles with either nodal tangent, it is clear that they touch the same confocal, and it follows from § 4 that they subtend equal or supplementary angles at any point of the curve.

viii. Hence, if p be the next point to P, the angle $OpP = OpQ = OPQ$ (fig. 19). Hence if the tangents at P, Q meet at R, OP bisects the angle RPQ, and similarly OQ bisects the angle RQP. Hence OR bisects the angle PRQ, *i.e.* OP, OQ subtend equal angles at R. Hence R is on the curve; or the two tangents from a point on the curve are equally inclined to the line joining the point to the node.

Let PQ meet the curve again at T, then since angle $PTO = QTO$, OT is perpendicular to PQ.

Hence O is the centre of one of the circles touching the sides of RPQ, and the point of contact of the circle and PQ lies on the curve.

The cubic is therefore also the locus of feet of perpendiculars from O on its polars with regard to the confocal family.

ix. The nodal tangent, since it makes equal angles with OP, OQ, also makes equal angles with OR, OT. For if PO be produced to p, $pOR = OPR + ORP =$ complement of $OQT = QOT$. Hence the tangents at RT meet on the curve at S, O is the centre of the circle inscribed in RST, and the foot of the perpendicular OH from O on TR is on the curve; the tangents at S and H meet on the curve, and so on. In fact in any cubic with a double point, if from a point R on the curve tangents RP, RQ are drawn, and PQ meets the curve again at T, the tangents at RT meet on the curve. For if the point be a crunode, then (Higher Plane Curves, § 215) any point may be taken $1 - \theta^2$, $\theta(1 - \theta^2)$, 1. Three point lie on a line if

$$\theta_2\theta_3 + \theta_3\theta_1 + \theta_1\theta_2 = -1,$$

and the tangent at θ meets the curve at ϕ if

$$\theta^2 + 2\theta\phi = -1.$$

Hence, if $\theta\theta'$ be points of contact of tangents from ϕ,

$$\theta+\theta'=-2\phi,\ \theta\theta'=1.$$

But if $\theta\theta'$ meet the curve again at ϕ',

$$\theta\theta'+\phi'(\theta+\theta')=-1,$$

therefore $$\phi\phi'=1.$$ Q. E. D.

So for the acnodal cubic.

x. The line through O and the middle point of any such chord as PQ passes through the centre of the confocals; and this line is a normal to the circle of infinite radius which is a limiting confocal, so that it is parallel to the real asymptote. Hence, if tangents be drawn from any point on the curve the locus of middle points of the chords of contact is a straight line parallel to the real asymptote.

The following construction gives the point A where the the asymptote cuts the curve.

Let M be the middle point of PQ; make angle $CON=BOM$. Then a line through O perpendicular to ON meets the asymptote on the curve (fig. 20). For in the triangle $A\infty N$. AN and $A\infty$ being tangents, ON, OM must be equally inclined to a nodal tangent, and since O is the centre of a touching circle the angle AON is right.

xi. Let OK be that normal to the loop which passes through O, and let the tangent at K meet the curve at L. Then L is the one real point of inflexion. For of the tangents drawn from L to the curve LK is one and the tangent at L, LB is the other. And since O is the centre of a circle touching the sides of the thin triangle KLL, OKL is a right angle.

January 8, 1887.

NOTE ON THE SINGULAR SOLUTIONS, &c., OF HOMOGENEOUS DIFFERENTIAL EQUATIONS.

By Prof. *W. Woolsey Johnson.*

If the homogeneous equation of the first order be solved for y, we have

$$y = x\phi(p) \text{ ....(1)},$$

whence

$$\frac{dp}{dx} = \frac{p - \phi(p)}{x\phi'(p)} \text{ ....(2)},$$

an equation which determines the curvature of the path of a moving point which satisfies the differential equation (1). If p_1 is a root of the equation $p - \phi(p) = 0$, we have $\frac{dp}{dx} = 0$ when $p = p_1$, and since $p = p_1$ in equation (1) gives

$$y = \phi(p_1)\, x = p_1 x,$$

this straight line is a solution of the differential equation. In fact the integral may be written in the form

$$f\left(\frac{x}{c}, \frac{y}{c}\right) = 0 \text{ ....(3)},$$

and represents a system of similar curves having the origin as a centre of similitude, and the line $y = p_1 x$ is a part of the particular integral corresponding to $c = 0$.

But if p_1 be at the same time a root of the equation $\phi'(p) = 0$, it is to be noticed that while at any point of the line $y = p_1 x$, $\frac{dp}{dx}$ still admits of the value zero which corresponds to the solution $y = p_1 x$, it is not this value but another which is consecutive with its values at neighbouring points not on this line, namely the value obtained by dividing out the factor $p - p_1$ from the two terms of the fraction in equation (2). This latter value accordingly determines the curvature of the particular integral, and $y = p_1 x$ is now a singular solution.

If α be a root of $\phi'(p) = 0$, which is not also a root of $p - \phi(p) = 0$, $\frac{dp}{dx}$ is infinite when $p = \alpha$. Hence $y = x\phi(\alpha)$ is a locus upon which the curvature is infinite, that is to say, a cusp-locus.

Differentiating (1) with respect to p, we have $x\phi'(p)=0$, from which it would seem that, α, β, &c., being the roots of $\phi'(p)=0$,

$$y=x\phi(\alpha),\quad y=x\phi(\beta),\quad \&c.,$$

should include the complete locus of equal roots for p, so that $\phi'(p)=0$ would be satisfied at the tac-locus. Moreover, since $\frac{dp}{dx}$ has two values at a point on the tac-locus for a single set of values of x, y and p, we might expect $\frac{dp}{dx}$ to take the indeterminate form at a tac-locus; but we have seen that the numerator vanishes only for a singular solution and for the particular integral corresponding to $c=0$.

The explanation depends upon the fact that $\phi(p)$ is not a one-valued function when a tac-locus exists. Suppose for example that the given homogeneous equation is of the second degree with respect to p, and also of the second degree with respect to $\frac{y}{x}$, which for the moment denote by z. Suppose also that for a given value of z the equation in p has a pair of roots equal to α, the equation is then divisible by $(p-\alpha)^2$. In the process of solving for y we have virtually resolved the equation into two irrational factors, thus

$$[y-x\phi_1(p)]\,[y-x\phi_2(p)]=0 \text{............} (4),$$

where $\phi_1(p)$ and $\phi_2(p)$ are the two values of $\phi(p)$ in equation (1). It may happen that one of these factors (say the second) does not vanish when $p=\alpha$; in that case, the other factor is of the form

$$y-x\phi_1(p)=(p-\alpha)^2 Q,$$

where Q is finite when $p=\alpha$, and the derivative of this factor vanishes when $p=\alpha$, so that α is a root of the equation $\phi_1'(p)=0$, that is to say, of the equation $\phi'(p)=0$, exactly as if $\phi(p)$ were a one-valued function. In this case, then, α determines a cusp-locus.

On the other hand it may happen that each of the factors of (4) vanishes when $x=\alpha$; in which case, each of the factors is of the form

$$y-x\phi_1(p)=(p-\alpha)\,Q,$$

and the derivative of neither vanishes when $x=\alpha$, so that α is *not* a root of $\phi'(p)=0$. In this case α determines a tac-locus. The manner in which two values of $\frac{dp}{dx}$ in equation (2) now arise is this: when $p=\alpha$, each of the values of $\phi(p)$ is equal to the given value of z, thus in the numerator either form of $\phi(p)$ may be used; hence either form of $\phi'(p)$ may be used in the denominator, but these have different values giving two values of $\frac{dp}{dx}$ for $p=\alpha$.

The following example illustrates the points enumerated above,

$$p^2(x^2-2xy)-2pxy+y^2-2xy=0 \;\ldots\ldots\ldots(5).$$

The integral is

$$x^2+y^2-2c(x+y)+c^2=0 \;\ldots\ldots\ldots\ldots\; (6),$$

representing the circles touching the two coordinate axes.

When solved for y, the differential equation is

$$\frac{y}{x}=1+p+p^2\pm(1+p)\sqrt{(1+p^2)}=\phi(p)\ldots\ldots(7),$$

whence $$p-\phi(p)=-(1+p^2)\mp(1+p)\sqrt{(1+p^2)}\ldots\;\ldots(8),$$

and $$\phi'(p)=1+2p\pm\frac{1+p+2p^2}{\sqrt{(1+p^2)}}\;\ldots\ldots\ldots\ldots\;(9).$$

The roots of the equation $p-\phi(p)=0$, are 0, ∞ and $\pm i$, and each of these gives a solution of which the last two constitute the particular integral $x^2+y^2=0$. The root $p=0$, which makes one value of $p-\phi(p)$ vanish in (8), makes the corresponding value of $\phi'(p)$ also vanish in (9), $y=0$ is a singular solution, $\frac{dp}{dx}=0$ being admissible in equation (2); but if we evalute the indeterminate form $\frac{p-\phi(p)}{x\phi'(p)}$, when $p=0$, we shall find the value $\frac{1}{x}$ which gives the curvature of the particular integral at the point of contact $(x, 0)$.

The values of $\frac{y}{x}$ or z which give p equal roots in (5) are 0, ∞ and 1, the corresponding values of p being 0, ∞ and -1; but, whereas, when $z=0$ in (7), $p=0$ corresponds to but one of the forms of $\phi(p)$; when $z=1$, $p=-1$ satisfies the equation with either of the forms of $\phi(p)$. Thus, in the latter case, while $p-\phi(p)$ has but a single value, viz., -2 we may use either form of $\phi(p)$, and therefore either of the values of $\phi'(p)$, which are, from (9), $-1\pm\sqrt{2}$: and we have for $\frac{dp}{dx}$ the two values $\frac{-2(1\pm\sqrt{2})}{x}$.

SQUARING $\rho\,\mathrm{dn}\,u$.

By *N. M. Ferrers.*

THE following is a method of squaring $\frac{2K}{\pi}\,\mathrm{dn}\,\frac{2Kx}{\pi}$, based upon the fact that $\frac{2K}{\pi}\left(\mathrm{dn}\,\frac{2Kx}{\pi}+k\,\mathrm{cn}\,\frac{2Kx}{\pi}\right)$ is the same function of $\sqrt{q}$ and x that $\frac{2K}{\pi}\,\mathrm{dn}\,\frac{2Kx}{\pi}$ is of q and $2x$.

Writing ρ for $\frac{2K}{\pi}$ and u for $\frac{2Kx}{\pi}$ we have (see p. 145)

$$k\rho\,\mathrm{sn}\,u=\Sigma_1^\infty\,\frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}}\sin(2n-1)\,x\;\ldots\ldots\ldots\;(1),$$

$$k\rho\,\mathrm{cn}\,u=\Sigma_1^\infty\,\frac{4q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}}\cos(2n-1)\,x\;\ldots\ldots\ldots\;(2),$$

$$\rho\,\mathrm{dn}\,u=1+\Sigma_1^\infty\,\frac{4q^n}{1+q^{2n}}\cos 2nx\ldots\ldots\ldots\ldots\ldots(3),$$

whence $$\rho\,(\mathrm{dn}\,u+k\,\mathrm{cn}\,u)=1+\Sigma_1^\infty\,\frac{4q^{\frac{1}{2}n}}{1+q^n}\cos nx\;\ldots\ldots\;(4),$$

in accordance with the statement made above.

Hence, if

$$\rho^2\,\mathrm{dn}^2u=f_0(q)+\Sigma_1^\infty f_n(q)\cos 2nx,$$

we see that

$$\rho^2\,(\mathrm{dn}\,u+k\,\mathrm{cn}\,u)^2=f_0(\sqrt{q})+\Sigma_1^\infty f_n(\sqrt{q})\cos nx.$$

But $$\rho^2\,(\mathrm{dn}\,u+k\,\mathrm{cn}\,u)^2=\rho^2\,(\mathrm{dn}^2u+k^2\,\mathrm{cn}^2u+2k\,\mathrm{cn}\,u\,\mathrm{dn}\,u)$$
$$=\rho^2\,(2\,\mathrm{dn}^2u-k'^2+2k\,\mathrm{cn}\,u\,\mathrm{dn}\,u)\ldots(5).$$

Now, differentiating (1) with respect to x, we get

$$k\rho^2\,\mathrm{cn}\,u\,\mathrm{dn}\,u=\Sigma_1^\infty\,(2n-1)\frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}}\cos(2n-1)\,x.$$

Hence

$$f_0(q)+\Sigma_1^\infty f_n(\sqrt{q})\cos nx=2\,\{f_0(\sqrt{q})+\Sigma_1\,f_n(q)\cos 2nx\}-\rho^2k'^2$$
$$+\,2\Sigma_1^\infty\,(2n-1)\,\frac{4q^{\frac{1}{2}(n-1)}}{1-q^{n-1}}\cos(2n-1)\,x.$$

Hence, if n be any odd integer,

$$f_n(\sqrt{q})=2n\frac{4q^{\frac{1}{2}n}}{1-q^n};$$

and, if n be any even integer,

$$f_n(\sqrt{q}) = 2f^{\frac{1}{2}n}(q);$$

whence, by successive reduction, we get

$$f_n(\sqrt{q}) = 2n\frac{4q^{\frac{1}{2}n}}{1-q^n};$$

n being any integer whatever,

and $$f_n(q) = 2n\frac{4q^n}{1-q^{2n}}.$$

Hence

$$\rho^2 \operatorname{dn}^2 u = f_n(q) + \Sigma_1^\infty \, 2n\frac{4q^n}{1-q^{2n}} \cos 2nx.$$

The value of $f_n(q)$ or $1 + \Sigma_1^\infty \frac{8q^{2n}}{(1+q^{2n})^2}$, may be determined in the usual manner.

NOTE ON THE BINOMIAL-THEOREM COEFFICIENTS.

By *A. Gutzmer*.

In connexion with a note on a "Theorem relating to the sum of selected binomial-theorem coefficients,' which Prof. P. G. Tait has published in vol. XIII., pp. 154, 155 (1884) of the *Messenger of Mathematics*, I beg to communicate the following elementary relations between the sums of binomial coefficients, which I had noticed in 1883, and of which I was reminded by the perusal of Prof. Tait's note.

If we denote, as usual, by $\binom{\nu}{\kappa}$ the binomial coefficient

$$\binom{\nu}{\kappa} = \frac{\nu(\nu-1)\ldots(\nu-\kappa+1)}{1.2\ldots\kappa},$$

where ν and κ are both entire numbers and $\nu \geqq \kappa$, it is well known that

$$\Sigma_{\kappa=0}^{\nu} \binom{\nu}{\kappa} = 2^\nu,$$

and that the coefficients of the binomial $(1+x)^\nu$ for the successive values $\nu = 1, 2, \ldots$ may be arranged in the following scheme:

$$\begin{array}{ccccccccccccl}
 & & & & & 1 & & 1 & & & & & \ldots\ldots 2^1, \\
 & & & & 1 & & 2 & & 1 & & & & \ldots\ldots 2^2, \\
 & & & 1 & & 3 & & 3 & & 1 & & & \ldots\ldots 2^3, \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \ldots\ldots 2^4, \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \ldots\ldots 2^5, \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \ldots\ldots 2^6, \\
\end{array}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

Now it is easy to prove the following simple formulæ:

$$(1) \quad \Sigma_1^n 2^\nu = 2^{n+1} - 2;$$

$$(2) \quad \Sigma_1^n 2^{p+\nu} = 2^p (2^{n+1} - 2);$$

$$(3) \quad \frac{\Sigma_1^n 2^{p+\nu}}{\Sigma_1^m 2^{p+\nu}} = \frac{2^p (2^{n+1} - 2)}{2^p (2^{m+1} - 2)} = \frac{2^{n+1} - 2}{2^{m+1} - 2};$$

$$(4) \quad \frac{\Sigma_1^n 2^{p+\nu}}{\Sigma_1^n 2^{q+\nu}} = \frac{2^p (2^{n+1} - 2)}{2^q (2^{n+1} - 2)} = 2^{p-q};$$

$$(5) \quad \Sigma_1^n 2^{2p+2\nu+1} = 2\, \Sigma_1^n 2^{2p+2\nu};$$

$$(6) \quad \frac{\Sigma_1^n 2^{p+\nu\kappa}}{\Sigma_1^m 2^{p+\nu\kappa}} = \frac{\Sigma_1^n 2^{\nu\kappa}}{\Sigma_1^m 2^{\nu\kappa}} = 2^{n-m};$$

$$(7) \quad \frac{\Sigma_1^n 2^{p+\nu\kappa}}{\Sigma_1^m 2^{p+\nu\kappa}} = 2^{p-q}.$$

The summation in all the above formulæ has reference to the letter ν.

Expressing the contents of these formulæ in words, we find, having regard to the scheme, the following relations between the sums of binomial coefficients:

(1) The sum of the binomial coefficients of the n *first* powers [always of the binomial $(1+x)^\nu$] is equal to the sum of the binomial coefficients of the $(n+1)^{\text{th}}$ power minus the first and the last coefficients (each being equal to 1).

(2) The sum of the binomial coefficients of n *successive* powers (*i.e.* of the $(p+1)^{\text{th}}$, $(p+2)^{\text{th}}$, ..., $(p+n)^{\text{th}}$ power) is equal to the product of the sum of the coefficients of the n *first* powers and that of the power which immediately precedes the power from which we started (*i.e.* the p^{th} power).

(3) If we form the sum of the binomial coefficients of n *successive* powers, and that of m *successive* powers, starting from the same power, their *ratio* is equal to the ratio of the sums of the coefficients of the n and m *first* powers respectively, and is therefore independent of the power from which we started.

(4) The ratio of the sum of the binomial coefficients of *n successive* powers and that of *n other successive* powers is equal to the ratio of the sums of the binomial coefficients of the powers immediately preceding the powers from which we started.

(5) The sum of the binomial coefficients of *n successive odd* powers is equal to *half* the sum of the coefficients of the *next n successive even* powers and *twice* that of the preceding *n successive even* powers.

The relation (3) is only a particular case of the following more general theorem:

(6) If we form the sum of the binomial coefficients of *n successive* powers, the exponents of which have a constant difference κ, and that of m such powers, beginning at the *same* power, their ratio is equal to the ratio of the sums of the binomial coefficients of the n and m *first* thus selected powers, and is therefore independent of the power from which we started.

In the same manner (4) and (5) are only particular cases of this relation:

(7) The ratio of the sum of the binomial coefficients of n powers, the exponents of which have a constant difference κ, and that of *n other* such powers is equal to the ratio of the sums of the coefficients of the powers from which we started; therefore the ratio is independent of the number n and the difference κ, and depends only upon the difference of the exponents of the first powers respectively.

These elementary theorems relating to selected sums of binomial coefficients can also be extended, in a similar way, to the case of polynomial coefficients, since we have only to deal with geometrical series.

The preceding results, although very elementary with respect to the geometrical series, may still be regarded as of some interest when stated with reference to the binomial-theorem coefficients.

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END OF VOL. XVI.

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